U.S. DEPARTMENT OF COMMERCE
Maurice H. Stans, Secretary

OFFICE OF TELECOMMUNICATIONS
Armig G. Kandoian, Director
INSTITUTE FOR TELECOMMUNICATION SCIENCES
Richard C. Kirby, Director

OT/ITS RESEARCH REPORT 7

A New Method for Predicting HF Ground Wave Attenuation Over Inhomogeneous, Irregular Terrain

R. H. OTT

INSTITUTE FOR TELECOMMUNICATION SCIENCES
BOULDER, COLORADO
January 1971

For sale by the Superintendent of Documents, U. S. Government Printing Office, Washington, D. C. 20402
Price 60 cents

Sponsor: This work was supported by the Advanced Research Projects Agency, ARPA Order Number 1458

Project Monitor: Mr. Alvin Van Every

TABLE OF CONTENTS

ABSTRACT

Several examples of the numerical evaluation of an integral equation for the calculation of the attenuation of a radio wave are given. These waves are assumed to be propagated over realistic, smoothly varying irregular, inhomogeneous terrain. Results for propagation over a cylindrical earth show an accuracy to 3-4 significant figures when compared with the classical residue series. Calculations for propagation over smooth mixed land-sea paths agree with classical methods. The applicability of the program to permit computation of propagation over terrain with smooth height variation is demonstrated by calculations of propagation over one and two Gaussian-shaped hills. The ability of the program to allow treatment of variations in both ground conductivity and height combined is illustrated by calculations of propagation from the sea up a sloping beach and by calculations of propagation over an island. This last example illustrates the importance of the terrain profile in mixed path calculations.

1. INTRODUCTION

Despite numerous attempts, a numerically feasible way to calculate the field strength of a radio wave propagating over realistic, smoothly varying, inhomogeneous terrain, has not yet been found. Hufford (1952) developed an integral equation for such propagation by using the free-space Green's function in Green's second identity and showed that his solution yielded the classical result for propagation over a smooth sphere. Berry (1967) succeeded in solving the equation numerically for vertically polarized radio waves, showing sample calculations up to 10 MHz. If the normalized surface impedance is not much smaller than 1, the numerical techniques are very inefficient, however, and round-off errors accumulate so fast that the results are not useful. For normal ground constants, this condition excludes all horizontally polarized waves and all vertically polarized waves above a few megahertz.

The method used in this paper is based on an elementary function that is closely related to the Sommerfeld flat earth attenuation function. This elementary function satisfies a scalar "parabolic" wave equation. The resulting integral equation is numerically feasible for both vertical and horizontal polarization and for normalized surface impedances in the HF band.

The problem to be solved is illustrated in figure 1, which shows a possible propagation path. The signal at the receiver is affected by the mean curvature of the earth, height profile along the path, and the change of surface impedance along the path. The changes may be abrupt (e.g., at a land sea boundary), or gradual (e.g., as the sea state, temperature, or salinity change). The problem of abrupt changes in surface impedance at smooth land-sea boundaries has been solved (Wait, 1964). Numerical results for changes in surface impedance have been calculated by Rosich (1968, 1970).

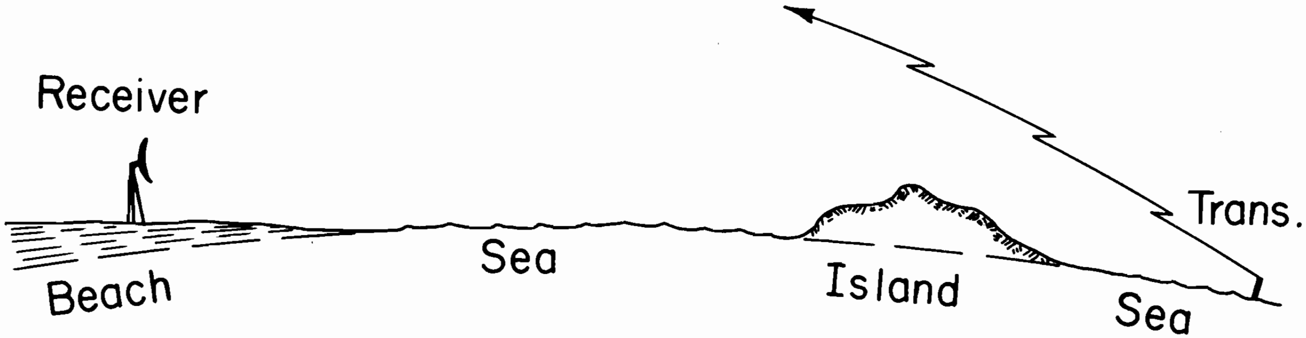


Figure 1. A possible propagation path.

The present work allows the terrain to be represented by a completely arbitrary profile in terms of the elevation versus distance. The hills and valleys themselves are taken to be uniform in the direction transverse to the propagation direction. The terrain may also be characterized by a conductivity and dielectric constant which are functions of distance.

The main body of the report describes the results of calculations for several examples including paths similar to that in figure 1. The appendices contain the derivation of the integral equation, the necessary numerical analysis, and a listing of the Fortran computer program.

2. THE INTEGRAL EQUATION

The derivation of the integral equation is given in appendix A. The details will not be reiterated here, but the final result is (Ott, 1971)

$$f(x) = g(x, y)\, W(x, 0) - \sqrt{\frac{i}{\lambda}} \int_0^x f(\xi)\, e^{-ik\omega(x,\xi)} \left\{ y'(\xi)\, W(x, \xi) - \frac{y(x) - y(\xi)}{x - \xi} + (\Delta - \Delta_r) \right\} \left[\frac{x}{\xi(x-\xi)} \right]^{\frac{1}{2}} d\xi \,, \qquad (1)$$

where x, ξ, $y(x)$ and $y(\xi)$ are defined in figure 2. The factor $(\Delta - \Delta_r)$ arises in mixed-path problems. That is, substituting Δ_r for Δ in (A-2) and (A-9) will yield the difference $(\Delta - \Delta_r)$. The factor Δ_r is constant with distance and is the relative value of the normalized surface impedance. This factor is computed using the values for σ and ϵ_r for the first section of a mixed path. The factor Δ varies with distance in a mixed path problem. The variation of Δ with x may be continuous or contain abrupt changes. The factor $(\Delta - \Delta_r)$ is zero for a single section path. The remaining factors in (1) are defined as

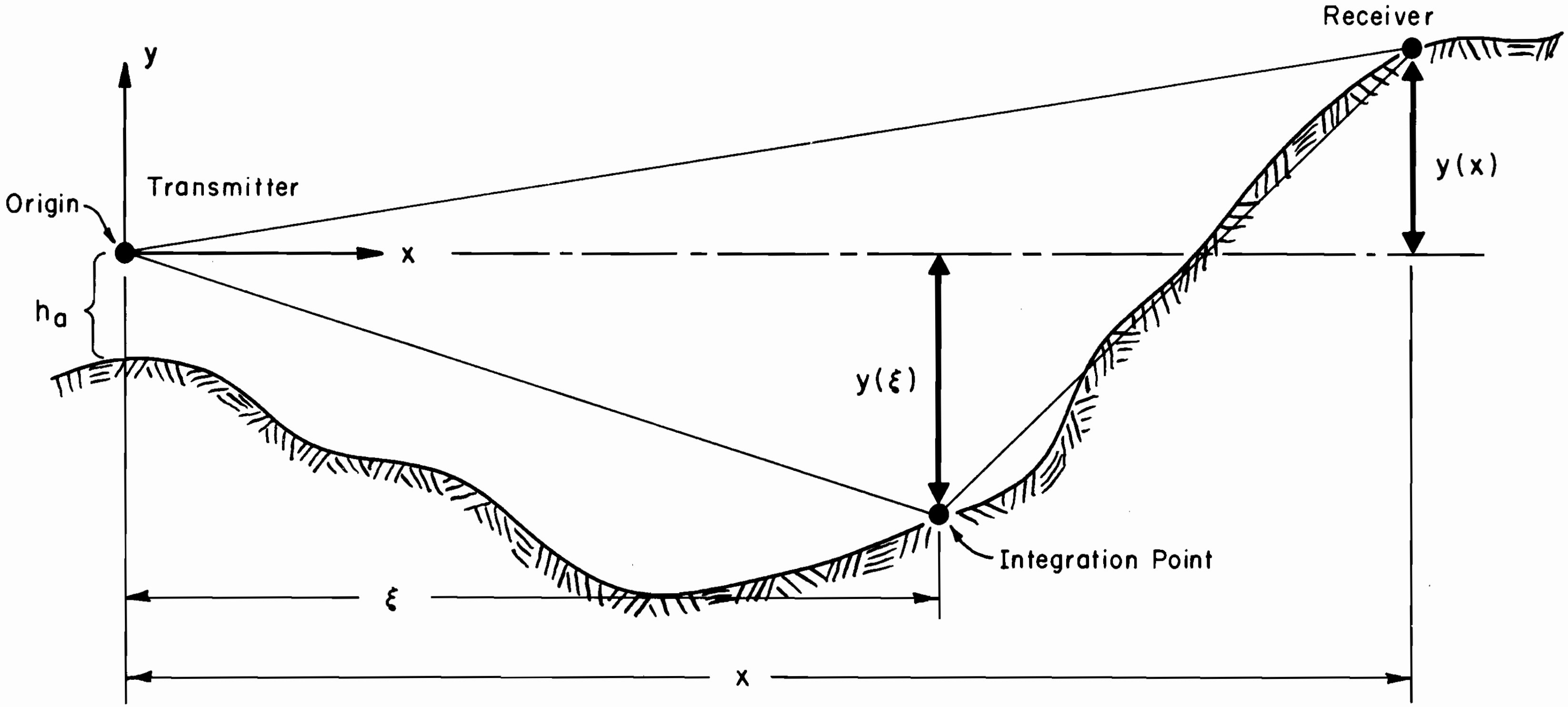


Figure 2. Geometry for integral equation

$$\omega(x, \xi) = \frac{[y(x) - y(\xi)]^2}{2(x - \xi)} + \frac{y^2(\xi)}{2\xi} - \frac{y^2(x)}{2x} ,$$

$$W(x, \xi) = 1 - i\sqrt{\pi p}\; w(-\sqrt{u}) ,$$

$$p = -ik\,\Delta^2(x - \xi)/2 ,$$

$$u = p\left\{1 - \frac{y(x) - y(\xi)}{\Delta(x - \xi)}\right\}^2 , \quad \xi < x$$

$$w(-\sqrt{u}) = e^{-u}\,\mathrm{erfc}\,(i\sqrt{u})$$

$$= \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{e^{-t^2}\,dt}{\sqrt{u} + t} \quad \text{(Abramowitz and Stegun, 1964)}$$

$$\Delta = \begin{cases} \dfrac{\sqrt{\eta - 1}}{\eta} , & \text{vertical polarization} \\ \sqrt{\eta - 1} , & \text{horizontal polarization} \end{cases}$$

$$\eta = \epsilon_r - \frac{i\,18(10^3)\,\sigma}{f(\mathrm{MHz})}$$

f = frequency, in MHz

σ = ground conductivity

ϵ_r = dielectric constant

$g(x, y)$ = antenna pattern factor.

Equation (1) gives the integral equation for the attenuation function normalized to twice the free-space field. The details of the numerical solution of (1) are given in appendix B. Since the upper limit of integration in (1) is x, the effects of backscatter are excluded. That is, to include the effects of backscatter, the range of integration would have to include the entire terrain. Also, the integral equation in (1) neglects the effects of side-scatter since the derivation of (1) assumed ridges uniform in the direction transverse to the propagation direction. In the case of small slopes and the transmitting antenna near the earth, side-scatter and backscatter are second order effects.

3. EXAMPLES

In this section we examine the behavior of the attenuation function, f(x), for eight terrain profiles, y(x). Comparisons of results from (1) with previous results for a flat earth, a smooth homogeneous cylindrical earth, a smooth sea-land-sea path and a single Gaussian-shaped ridge seem to validate the technique. Its more general applicability is illustrated by calculations for propagation over two Gaussian hills, over an island that rises above sea level, and over a sea-sloping beach with a sand-dune path.

3.1 A Flat Earth

$y(x) = 0$, $y'(x) = 0$. The solution of the integral equation (1) is trivial and is simply

$$f(x) = W(x), \tag{2}$$

where $W(x)$ is the Sommerfeld flat-earth attenuation function (Wait, 1964).

3.2 A Parabodoidal Earth

$y(x) \simeq -x^2/2a$, $y'(x) = -x/a$, where a is the radius of the cylinder and is taken to be about 6.37×10^3 kilometers. The frequency of the transmitting antenna is 1 MHz and is vertically polarized. The ground constants are: $\sigma = 0.01$ mho/m and $\epsilon_r = 10$. The magnitude and phase of the attenuation function versus horizontal distance, x are given in table I. These results are compared in table I with those obtained using the residue series (Wait, 1964) for the attenuation function. The agreement is seen to be very good out to the largest distance computed. For example, at 300 km, the difference in the phase between the integral equation method and the residue series method is about 0.009 rad. or about 0.5°. The greatest error in amplitude occurs at a distance of 150 km and is about 2 units in the third significant figure. The error decreases on either side of this point, a characteristic common to many numerical solutions. The results obtained in table I are for a step size, h = 1 km; however, a step size of 2 km did not change the results appreciably. A detailed error analysis is beyond the scope of this paper. The last significant figure of agreement in table I is underlined.

Table I. Attenuation function versus distance

Horizontal Distance, x (km.)	Integral Equation Solution: Amplitude	Integral Equation Solution: Phase (rad.)	Residue Series Solution: Amplitude	Residue Series Solution: Phase (rad.)
0	1.0	0	1.0	0
25	0.51331	-1.9717	0.51332	-1.9709
50	0.28936	-2.5929	0.28970	-2.5921
75	0.17575	-2.9597	0.17595	-2.9556
100	0.11506	3.0902	0.11520	3.0892
125	0.08044	2.9126	0.08000	2.9131
150	0.05914	2.7606	0.05939	2.7663
175	0.04504	2.6169	0.04502	2.6120
200	0.03512	2.4736	0.03509	2.4680
225	0.02781	2.3278	0.02777	2.3213
250	0.02224	2.1780	0.02221	2.1710
275	0.01790	2.0249	0.01788	2.0168
300	0.01447	1.8681	0.01446	1.8591

The time required to compute the attenuation function at intervals of one kilometer out to a maximum distance of 300 km was about 25 min using a CDC 3800 computer. The time required to compute the attenuation function for a specified profile is approximately proportional to the square of the number of points used along the abscissa. Thus, in the above example, if the maximum distance were 150 km rather than 300 km, the time required would be about 1/4 as much, or about 6 min. The sample input and output data given in appendix C pertain to this sample.

3.3 A Gaussian-Shaped Ridge

$y = e^{-(x-5)^2}$, $y' = -2(x-5)y$. This is a more interesting profile at least from the standpoint of radio propagation. The profile together with the magnitude of the attenuation function versus distance are shown plotted in figure 3. The magnitude of the attenuation function $|f(x)|$, is normalized to twice the free space field, $2\exp(-ikr_o)/r_o$. The observer is located on the terrain and the transmitter is located at the coordinate origin. The ground constants are $\sigma = 0.01$ mho/m and $\epsilon_r = 10$. The transmitter is vertically polarized and the frequency is 1 MHz. The terrain profile shown in the insert has a maximum height of 1 km and the hill is centered at a point 5 km from the transmitter. The solid straight line in figure 3 is the attenuation function, $W(x)$, for a flat earth.

The data in figure 3 represented by crosses was obtained by replacing the Gaussian-shaped ridge with a rounded knife-edge and computing the field on the surface shown dashed in figure 4 using "4-ray theory" (Schelleng, et. al., 1933). The radius of the rounded knife-edge is 500 m (which is the curvature of the Gaussian

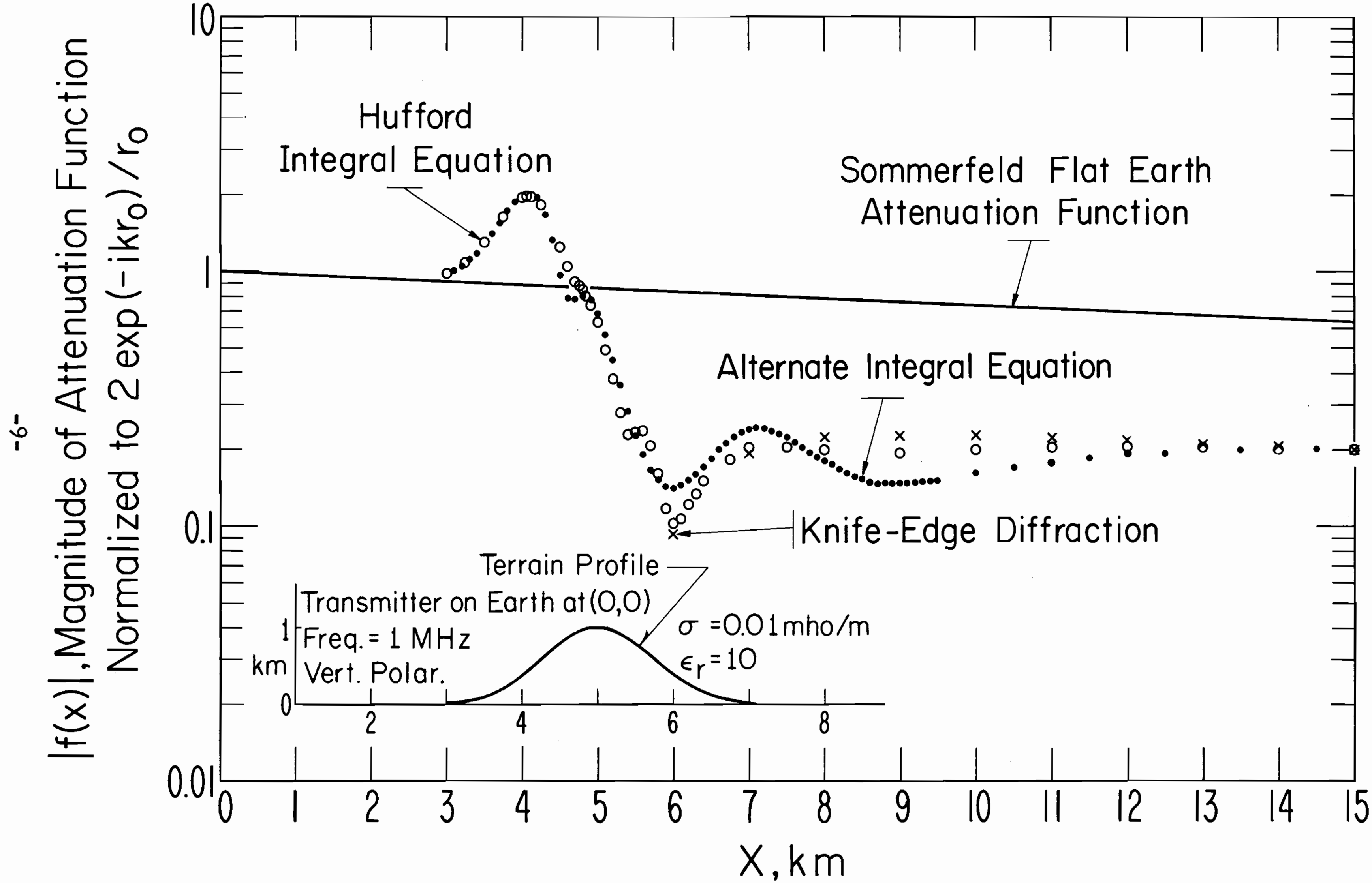


Figure 3. Propagation over a Gaussian-shaped ridge. Terrain profile is shown in insert. Observation point (receiving antenna) is located on the profile.

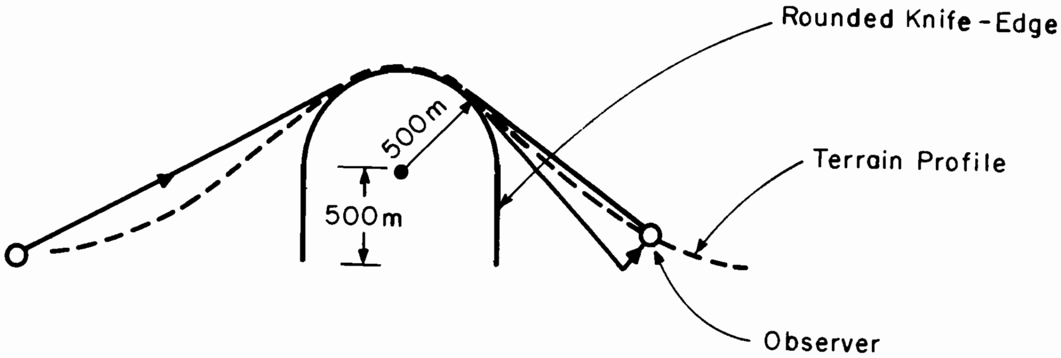


Figure 4. Rounded knife-edge approximation used to analyze the Gaussian-shaped ridge.

hill at its crest) and the knife-edge is located 1 km above the plane $y = 0$. The four rays are the two rays that strike the knife-edge on the illuminated side plus the two rays that reach an observer in the shaded side i.e., a direct diffracted ray and a ray which is diffracted and then reflected before reaching the observer. The results in figure 3 show excellent agreement between the points computed using "4-ray theory" and those obtained solving the integral equation numerically.

The open circles in figure 3 were computed using the Hufford integral equation (Hufford, 1952). Since there are fewer approximations in the Hufford integral equation than in the results presented in this paper, the former should be considered the most accurate. Hufford's integral equation shows a slight dip in the attenuation function at a distance of about 9 km which is exaggerated by the solid circles but does not appear in the knife-edge results. Also, the open circles differ somewhat in the shadow from the results presented earlier by Berry (1967). There were projection factors, of the form $\sqrt{1 + (y')^2}$, omitted from Berry's results since in most applications these factors are nearly unity, i.e., y' is small. However, in the present example these factors become important.

The solid circles in figure 3 present the attenuation function computed numerically using the integral equation in (1). We find some error in the results obtained using the integral equation presented in this paper around 6 km and 9 km. However, the error is small and is exaggerated in this particular example because of the large slopes encountered on the terrain profile. The error is a result of the assumption that

$$\frac{\partial^2 \psi}{\partial x^2} \cong 0 \quad ,$$

or that the fast phase variation of φ with x is in the term $\exp(-ikx)$. In most terrain profiles, this will indeed be a good approximation and, in fact, in the present example yields adequate accuracy.

The physical characteristics of the results in figure 3 are interpreted most easily using the ray picture. The attenuation function decreases at the flat earth rate for the first $2\frac{1}{2}$ km. Then, as the observation point moves up the crest of the hill, the attenuation function increases due to focusing of the direct ray and the surface ray on the lit side of the crest. The attenuation function reaches its maximum value very close to the point on the terrain where there is an inflection point. This increase in the amplitude to a maximum on the lit side near the crest has also been predicted analytically by Wait and Murphy (1958). Just over the top of the hill the attenuation function decreases since the direct ray is no longer present and then the attenuation function partially recovers again due to the constructive interference of a direct diffracted ray and a diffracted ray traveling along the surface before reaching the observer.

3.4 A Sea-Land-Sea Path

The terrain profile is flat in this example and the ground constants change abruptly at the sea-land, land sea interfaces. This example was selected as a check on the mixed path capabilities of the method. The results for the magnitude of the attenuation function normalized to twice the free space field are plotted in figure 5 versus distance from the antenna in km. The antenna is vertically polarized and the frequency is 10 MHz. The solid circles represent the attenuation function computed numerically using (1). The open circles in figure 5 represent the attenuation function computed by Rosich (1968, 1970) using a perturbation approach. The data given by the crosses in figure 5 represents the attenuation function computed using a method based upon the classical residue series (Furutsu, et. al., 1964). This method is equivalent to that of Wait (1964). This latter method makes the fewest approximations for the three section earth considered in

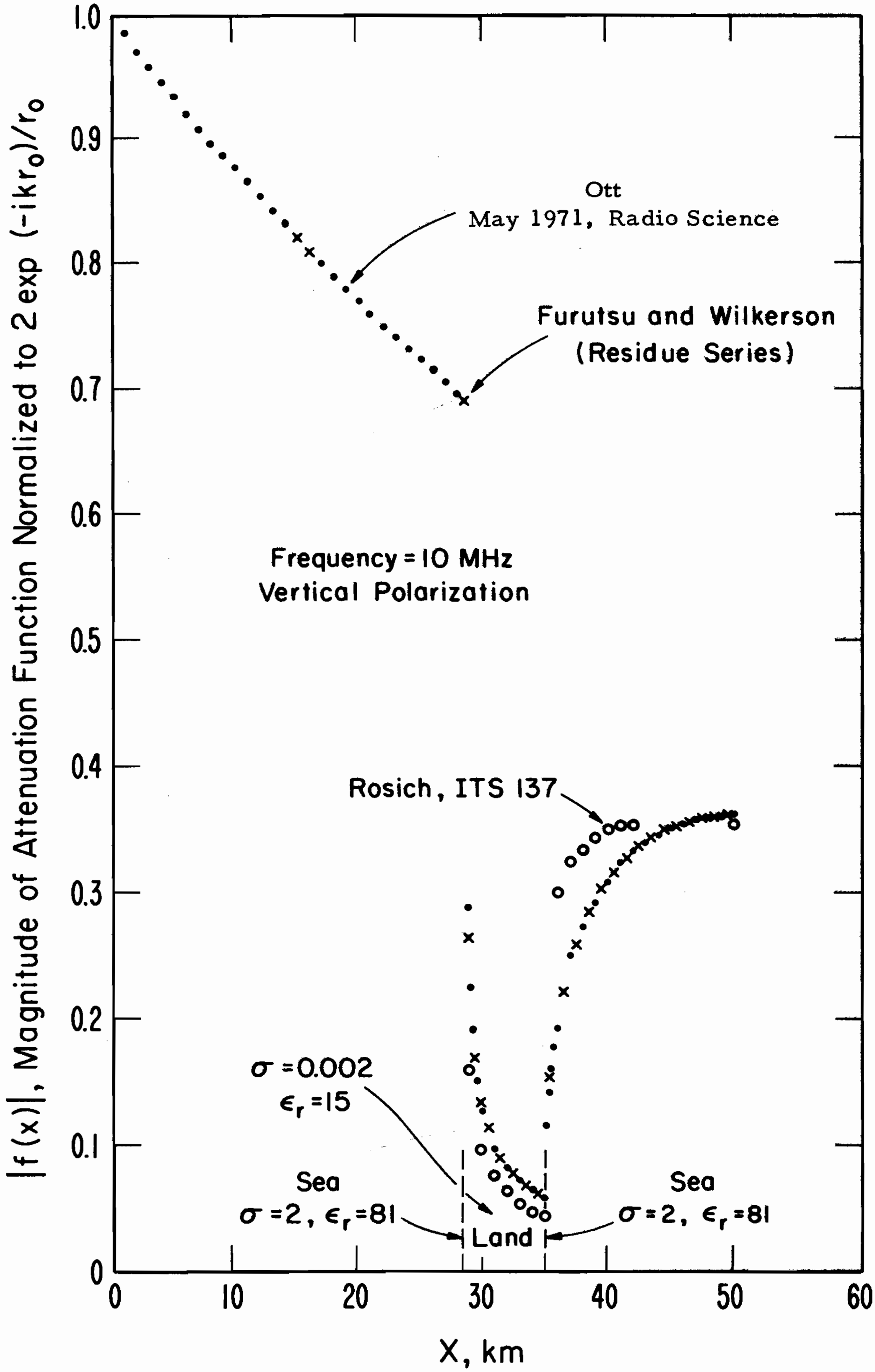


Figure 5. A sea-land-sea path. The profile is flat.

this example. The agreement between the solid circles representing (1) and the crosses, appears to demonstrate the validity of the formulation in treating mixed path propagation problems. The abrupt changes in conductivity and dielectric constant used in this example do not represent a realistic sea-land interface. The method is, however, capable of treating a continuous variation of conductivity and dielectric constant.

3.5 A Sea-Land-Sea Path With An Island

This example combines terrain features and mixed-path effects. The island is drawn to scale in figure 6 and its elevation is 250 m at the highest point. The magnitude of the attenuation function normalized to twice the free space field versus distance is plotted in figure 6. The antenna is vertically polarized and the frequency is 10 MHz. For comparison, the magnitude of the attenuation function for a flat island is also shown in figure 6. The most significant feature of figure 6 is that the terrain profile has a greater effect on the attenuation function on the island than do changes in the ground constants, and the residual effect of the profile well beyond the island is comparable to that of the change in ground constants.

3.6 A Sloping Beach At High And Low Tides

The profile is drawn to scale in figure 7 and the assumed ground constants used for the wet and dry sand are given in the figure. The transmitter is out at sea. As the tide rises, the wet sand in figure 7 is covered by water and as the tide recedes it exposes the wet sand. The magnitude of the attenuation function versus distance is shown plotted in figure 7. There is little difference in the attenuation function at high and low tide. However, the presence of the crest in the beach produces a peak in the attenuation function on the lit side and a shadow in back. This illustrates the importance of the terrain profile in mixed path problems.

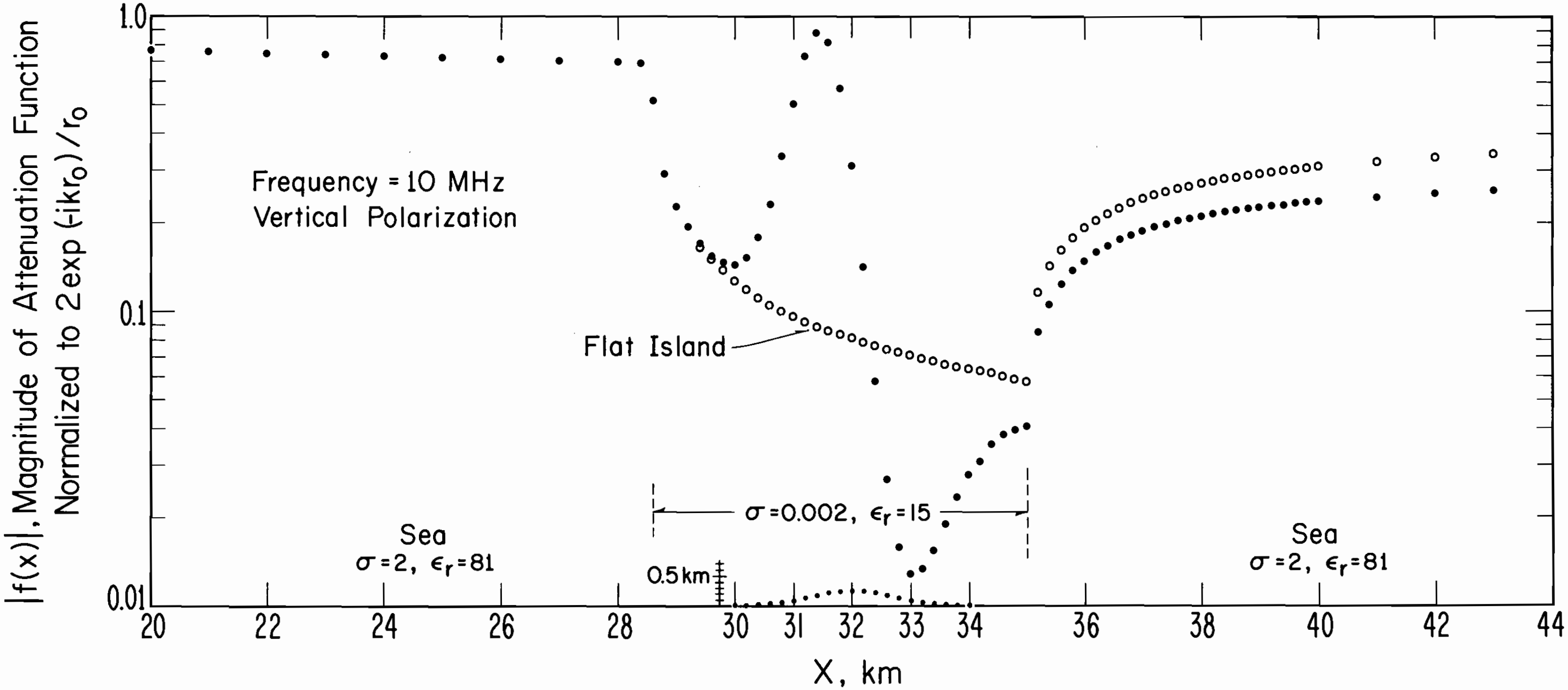


Figure 6. A sea-land-sea path with an island. The island is shown to scale in the insert.

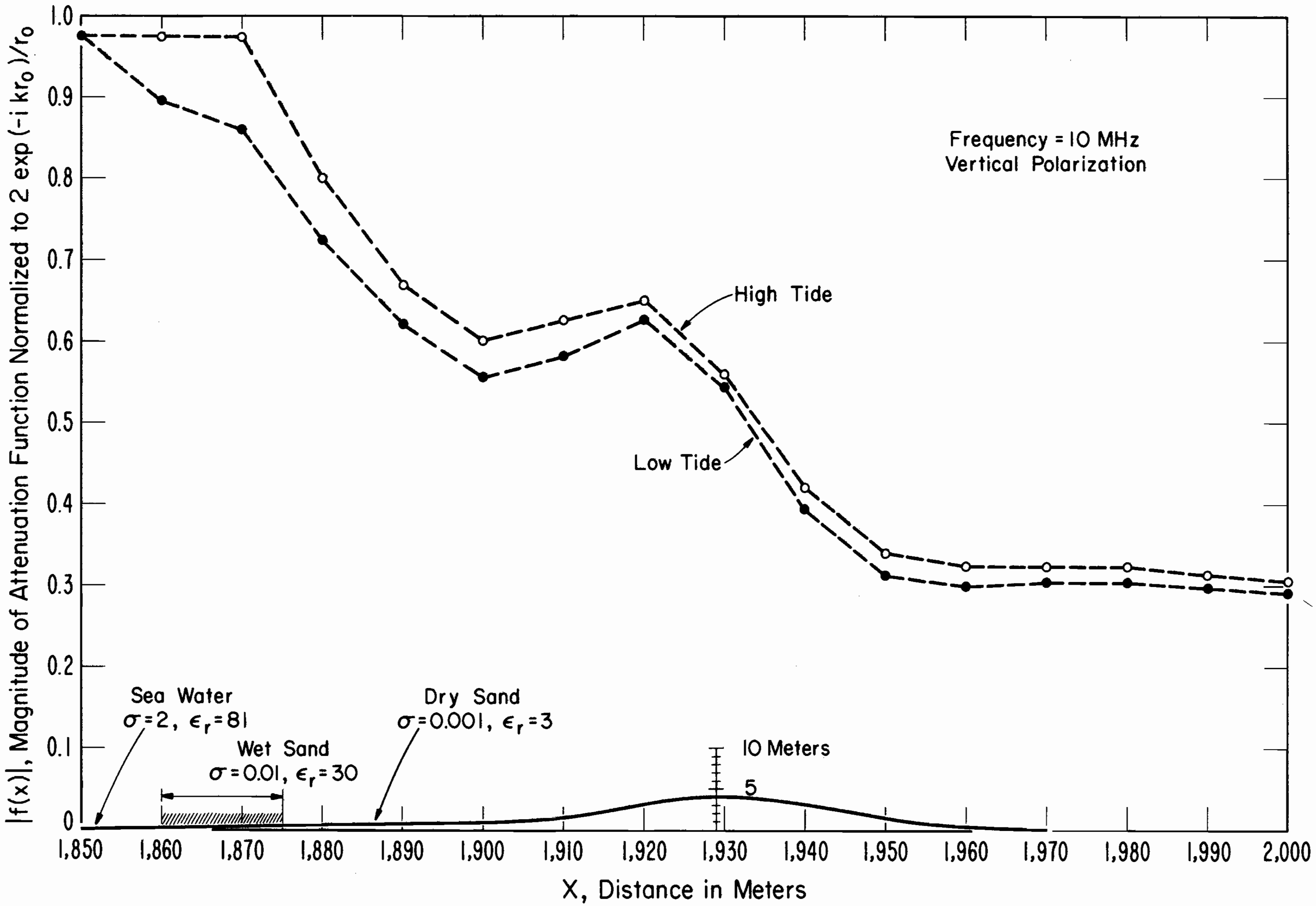


Figure 7. A sloping beach at high and low tides.

3.7 Two Gaussian Hills

The profile is drawn to scale in figure 8. The separation of the hills is such that a null instead of a peak in the attenuation function is produced on the lit side of the second hill. Obviously there are an infinite number of combinations of hills that will in turn produce an infinite number of possible combinations of nulls and peaks in the attenuation function. The method will, in principle, treat any number of hills and valleys. The hills need not have Gaussian profiles; any smooth function of distance is acceptable.

3.8 A Gaussian Hill (transmitting frequency of 10 MHz)

The profile as well as the magnitude of the attenuation function versus distance is shown in figure 9. The results in figure 9 differ somewhat from those published earlier by Berry (1967). Near the crest of the hill small oscillations in the attenuation occur which were not present when the transmitting frequency was 1 MHz. One possible explanation for these wiggles is numerical instability. However, this explanation was discarded when finer subdivisions of the integration interval failed to remove the oscillations. At present, they can only be explained in terms of an interference effect between a ground-reflected wave and the ground wave the former being stronger at 10 MHz than at 1 MHz. This case represents a quasi upper limit in the capability of the computer program in terms of frequency and slopes. That is, higher frequencies can be treated but the terrain cannot change as fast as it does in figure 9. Conversely, more rapid changes in terrain can be treated provided the frequency is less than it is in figure 9. Since the slopes in figure 9 are near unity, we have a heuristic uncertainty principle for our computer program

$$y' \; f \leq 10 \text{ (MHz)} .$$

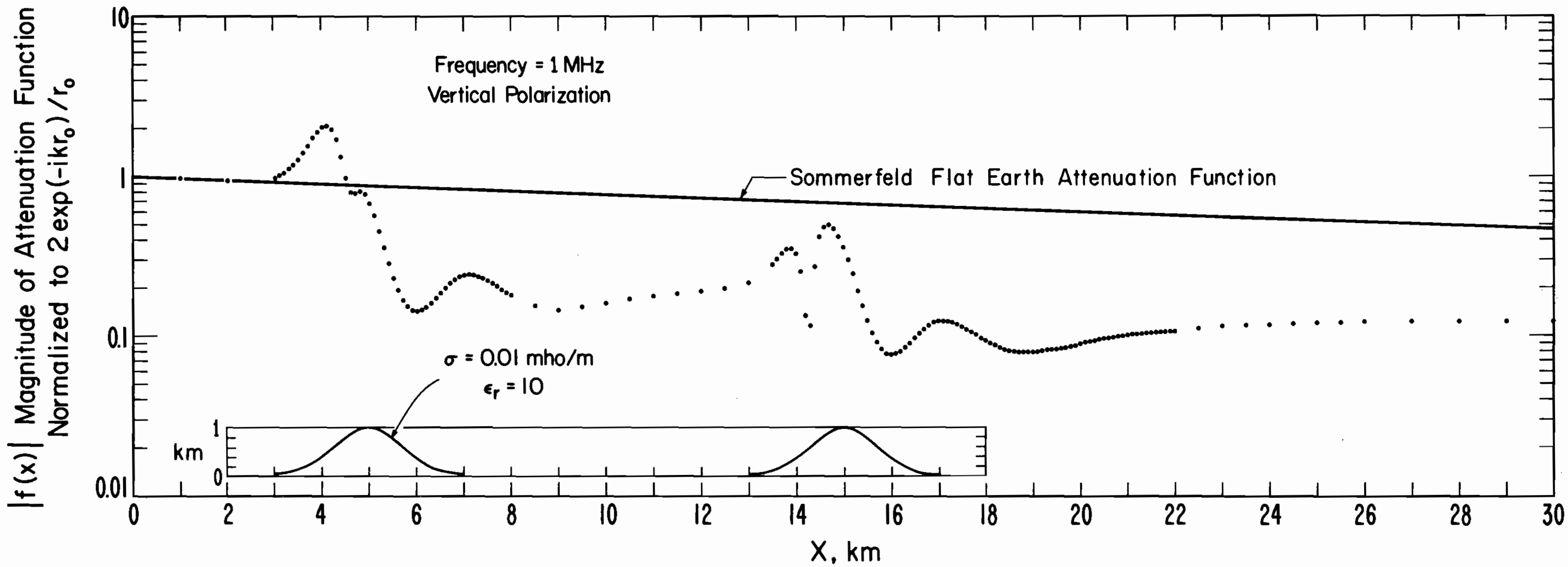


Figure 8. Two Gaussian-shaped ridges.

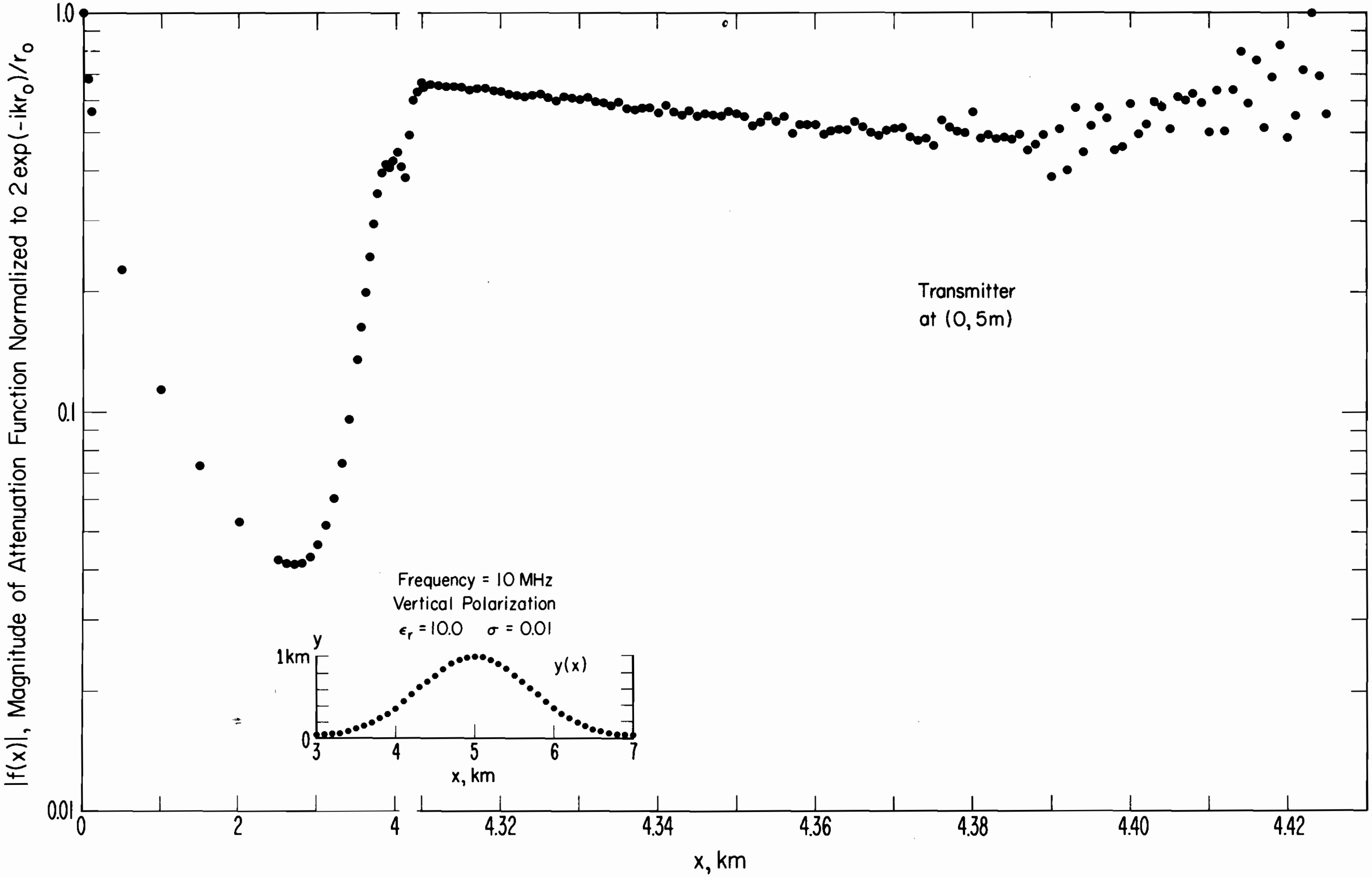


Figure 9. A Gaussian-shaped ridge at 10 MHz.

4. RECOMMENDATIONS AND CONCLUSIONS

The numerical evaluation of an integral equation for the propagation of radio waves over irregular, inhomogeneous terrain is demonstrated for several examples. Some of the examples provide a realistic picture of the attenuation of a radio wave when it encounters a terrain anomaly, such as a large conducting ridge. The Gaussian-Hill example at 1 MHz yields physical insight into a focusing phenomenon of the field just before the crest of a hill that cannot be predicted on the basis of simple diffraction theory, but is in fact predicted by the numerical solution of the integral equation. However, ray theory in a concave region with multiple reflections may work.

It appears that the results discussed in this report represent a useful tool for analyzing the attenuation loss of a radio wave as it encounters terrain anomalies such as hills, valleys, land-sea interfaces, etc. The computer program for this analysis is listed in appendix C. However, there are improvements that should be studied. They are listed below in an order not necessarily representing their relative importance.

1) A three-dimensional model of the terrain. It should be determined if the energy follows a geodesic and if the effects of transverse curvature are important or not.

2) Since the solution represented by the integral equations does in fact represent a solution of the wave equation plus boundary conditions, it applies to VHF frequencies as well as HF frequencies. Consequently, numerical techniques should be studied so that the program will handle VHF frequencies efficiently.

3) Real antennas rather than an idealized point source with an arbitrary pattern factor should be investigated; especially when a large diffracting obstacle is within the first Fresnel zone of the antenna.

5. ACKNOWLEDGMENTS

I thank Mr. Alvin Van Every at the Advanced Research Projects Agency for sponsoring this research. I also wish to thank Mrs. J. Stephenson for her assistance in the development of the computer program.

6. REFERENCES

Abramowitz, M., and Stegun, I. A., (1964), Handbook of Mathematical Functions, NBS, Applied Mathematics Series 55, p. 297.

Berry, L. A. (1967), "Radio propagation over a gaussian-shaped ridge," IEEE Trans. Ant. and Prop., AP-15, No. 5, pp. 701-702.

Furutsu, K., Wilkerson, R. E., and Hartmann, R. F., (1964), "Some numerical results based on the theory of radio wave propagation over inhomogeneous earth," Radio Science, J. Res., NBS 68D, No. 7.

Hufford, G. A. (1952), "An integral equation approach to the problem of wave propagation over an irregular terrain," Quart. Appl. Math., 9, pp. 391-404.

Ott, R. H. (1971), "An alternative integral equation for propagation over irregular terrain, Part II," to be published Radio Science, May.

Rosich, R. K. (1968), High-frequency ground-wave attenuation over inhomogeneous paths (unpublished)*, ESSA Tech. Memo ERLTM-ITS 137.

Rosich, R. K. (1970), Attenuation of high-frequency ground-waves over an inhomogeneous earth, U. S. Dept. Commerce, Office of Telecommunications, Telecommunications Research Report (to be published).

Schelleng, J. C., Burrows, C. R., and Ferrell, E. B. (1933), "Ultra-short wave propagation," Proc. IRE, 21, No. 3, pp. 427-463.

* This document is in the public domain, but it is considered unpublished since it was not printed for wide public distribution. This document is now out of print.

Wait, J. R. (1964), Electromagnetic surface waves, Advances in radio research, Ed. J. A. Saxton, 1, 157-217 (Academic Press, London and New York).

Wait, J. R., and Murphy, A. (1958), "Further studies of the influence of a ridge on the low-frequency ground wave," J. Res. NBS, 61D, pp. 57-60.

APPENDIX A: Derivation of the integral equation

Consider a solution, φ, of the wave equation

$$\text{(i)} \quad \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + k^2 \varphi = -2\pi \tau(x, y) \quad , \quad y > y(x)$$

which satisfies an impedance boundary condition of the form

$$\text{(ii)} \quad \frac{\partial \varphi}{\partial n} = \frac{ik\Delta\varphi}{\sqrt{1 + (y')^2}} \quad , \quad y = y(x)$$

where φ represents the vertical component of $\underline{E}$ for the case of vertical polarization or the vertical component of $\underline{H}$ for horizontal polarization. The time dependence is $\exp(i\omega t)$ and the normalized impedance, Δ, near grazing is

$$\Delta = \begin{cases} \dfrac{\sqrt{\eta - 1}}{\eta} \quad , & \text{vertical polarization} \\[2ex] \sqrt{\eta - 1} \quad , & \text{horizontal polarization} \end{cases}$$

with

$$\eta = \epsilon_r - \frac{i\sigma}{\omega \epsilon_o}$$

where ϵ_r is the dielectric constant, σ is the conductivity and ω the angular frequency.

The source distribution is $\tau(x, y)$. Let

$$\varphi = e^{-ikx} \psi(x, y)$$

and i) becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - 2ik\frac{\partial \psi}{\partial x} = -2\pi\tau\,(x,y)\;e^{ikx}$$

Assuming that the fast variation with x occurs in $\exp(-ikx)$

$$\frac{\partial^2 \psi}{\partial x^2} \cong 0$$

or that $\partial^2 \psi / \partial x^2$ is small compared with remaining terms we find

$$\frac{\partial^2 \psi}{\partial y^2} - 2ik\frac{\partial \psi}{\partial x} = -2\pi\tau(x,y)\;e^{ikx} \qquad \text{(A-1)}$$

An elementary function for (A-1) is (Ott and Berry, 1970)

$$\sqrt{\frac{2ik}{\pi}}\,G(x,y;\xi,\eta) = \frac{e^{-ik(\eta-y)^2/2(\xi-x)}}{\sqrt{\xi-x}}$$

$$+ \frac{ik\Delta\, e^{-ik\Delta\eta}}{\sqrt{\xi-x}} \int_{\eta}^{\infty} \exp\{-ik(t-y)^2/2(\xi-x)\}\,e^{ik\Delta t}\,dt,\; x<\xi$$

$$= \frac{e^{-ik(\eta-y)^2/2(\xi-x)}}{\sqrt{\xi-x}} \quad W(x,\xi),\; x<\xi.$$

$$\sqrt{\frac{2ik}{\pi}}\,G(x,y;\xi,\eta) = 0,\; x>\xi\,.$$

The function satisfies

$$\frac{\partial^2 G}{\partial y^2} + 2ik\frac{\partial G}{\partial x} = -2\pi\delta(x - \xi, y - \eta) \qquad \text{(A-3)}$$

The constant on the left-hand-side of (A-2) comes from integrating both sides of (A-3) over the region $R = \{x, y: -\infty < x \leq \infty, y(x) < y < \infty\}$.

Multiplying (A-1) by G, (A-3) by ψ, and subtracting and integrating over the region R yields

$$\iint_R \left(G\frac{\partial^2 \psi}{\partial y^2} - \psi\frac{\partial^2 G}{\partial y^2}\right) dx\, dy - 2ik\iint_R \left(G\frac{\partial \psi}{\partial x} + \psi\frac{\partial G}{\partial x}\right) dx\, dy$$

$$= -2\pi\iint_\Sigma e^{ikx}\,\tau\, G\, dx\, dy + \pi\psi(P) \qquad \text{(A-4)}$$

where P is the observation point (ξ, η), and Σ is a region around the source. The divergence theorem yields on the surface $y(x)$

$$\iint_R \left(G\frac{\partial^2 \psi}{\partial y^2} - \psi\frac{\partial^2 G}{\partial y^2}\right) dx\, dy = \int_C \left(G\frac{\partial \psi}{\partial y} - \psi\frac{\partial G}{\partial y}\right) \underline{e}_n \cdot \underline{e}_y\, dc \qquad \text{(A-5)}$$

where $\underline{e}_n$ is the outward directed normal (into the surface) and C is a contour enclosing R and

$$\underline{e}_n = -\frac{-y'\underline{e}_x + \underline{e}_y}{\sqrt{1 + (y')^2}}$$

and along $y = y(x)$

$$dc = \sqrt{1 + (y')^2}\ dx$$

Also

$$2ik \iint_R \left(G \frac{\partial\psi}{\partial x} + \psi \frac{\partial G}{\partial x}\right) dx\,dy = 2ik \iint_R \frac{\partial}{\partial x} (G\psi)\, dx\,dy$$

$$= 2ik \int_C G\,\psi\, \underline{e}_n \cdot \underline{e}_x\, dc \qquad \text{(A-6)}$$

From (ii), and neglecting $\partial\psi/\partial x$ compared with other terms

$$\frac{\partial\psi}{\partial y} = ik\Delta\psi - iky'\,\psi \qquad \text{(A-7)}$$

and substituting (A-5), (A-6), and (A-7) into (A-4), and assuming $\psi = 0$, for $x \le 0$, which means neglecting backscatter from the region $x \le 0$, and all sources are in the region $x > 0$,

$$-\int_0^{\xi} \left[ik\Delta\psi G - iky'\psi G - \psi \frac{\partial G}{\partial y}\right] dx - 2ik \int_0^{\xi} G\,\psi\, y'\, dx$$

$$+ 2\pi \iint_{\Sigma} \tau\, e^{ikx} G\, dx\,dy = \pi\,\psi\,(P)$$

or

$$\int_0^{\xi} [-ik\Delta\psi G - iky'\psi G + \psi \frac{\partial G}{\partial y}]\, dx + 2\pi \iint_{\Sigma} \tau G e^{ikx}\, dx\, dy = \pi\psi(P) \qquad \text{(A-8)}$$

Substituting (Ott and Berry, 1970)

$$\sqrt{\frac{2ik}{\pi}}\frac{\partial G}{\partial y} = ik\Delta\sqrt{\frac{2ik}{\pi}}\, G + \frac{ik\exp\{-ik(\eta-y)^2/2(\xi-x)\}}{\sqrt{\xi-x}}\left[\frac{\eta-y}{\xi-x}\right] \qquad \text{(A-9)}$$

in (A-8) gives

$$-ik\int_0^{\xi}\{y'\psi G - \psi\frac{\exp\{-ik(\eta-y)^2/2(\xi-x)\}}{\sqrt{\xi-x}}\left[\frac{\eta-y}{\xi-x}\right]\}\, dx$$

$$+ 2\pi\iint_{\Sigma} \tau G e^{ikx}\, dx\, dy = \pi\psi(P) \, .$$

Reintroducing φ and defining $\hat{G} = G e^{-ik(\xi-x)}$ yields

$$-\frac{ik}{2\pi}\int_0^{\xi}\{y'\varphi\hat{G} - \varphi\frac{\exp - ik\{(\xi-x) + [(\eta-y)^2/2(\xi-x)]\}}{\sqrt{\xi-x}}\left[\frac{\eta-y}{\xi-x}\right]\}\, dx$$

$$+ \iint_{\Sigma} \tau\hat{G}\, dx\, dy = \frac{1}{2}\varphi(P)e^{-ik\xi} \qquad \text{(A-10)}$$

We assume that the antenna has a phase center where the source distribution, $\tau(x, y)$, is located. Then we write

$$\tau(P) = g(P)\left\{\exp\left[-ik\left(x + \frac{y^2}{2x}\right)\right] \Big/ \sqrt{x}\right\} \delta(x, y) \qquad \text{(A-11)}$$

where $(x + y^2 / 2x)$ is the first two terms in the binomial expansion for the distance between the source point O and the observation point at P. The function $g(P)$ represents the antenna gain or pattern factor. We also introduce an attenuation function $f(P)$ defined as

$$\varphi(P) = 2 f(P) \exp\left[-ik\left(x + \frac{y^2}{2x}\right)\right] \Big/ \sqrt{x} \qquad \text{(A-12)}$$

When these two equations are substituted into (A-10), we find (interchanging (ξ, η) with (x, y))

$$f(x) = g(x, y)\, W(x, o)$$

$$- \sqrt{\frac{ik}{2\pi}} \int_o^x f(\xi)\, e^{-ik\omega(x, \xi)} \left\{ y'(\xi)\, W(x, \xi) - \frac{y-\eta}{x-\xi} \right\} \left[\frac{x}{\xi(x-\xi)} \right]^{\frac{1}{2}} d\xi$$

(A-13)

where

$$\omega(x, \xi) = \frac{(y-\eta)^2}{2(x-\xi)} + \frac{\eta^2}{2\xi} - \frac{y^2}{2x}$$

$$y = y(x)$$

$$\eta = y(\xi)$$

which differs slightly from the result in Ott and Berry (1970); see for example Ott (1971).

$$W(x,\xi) = 1 - i\sqrt{\pi p}\; e^{-u}\; \mathrm{erfc}(i u^{\frac{1}{2}})$$

$$p = \frac{-ik\Delta^2 (x-\xi)}{2} \qquad \text{(A-14)}$$

$$u = p\left\{1 - \frac{y-\eta}{\Delta(x-\xi)}\right\}^2 \quad , \quad \xi < x$$

References

(A-1) Ott, R. H. and Berry, L. A. (1970), "An alternative integral equation for propagation over irregular terrain," Radio Science, 5, No. 5, pp. 767-771.

(A-2) Ott, R. H. (1971), "An alternative integral equation for propagation over irregular terrain, Part II," to be published Radio Science, May.

APPENDIX B: Numerical Analysis

The integral equation (1) or equivalently (A-13) is of the form of a linear Volterra integral equation of the second kind, i.e.,

$$f(x) = g(x) - c \int_{0}^{x} f(s)\, K(x, s)\, ds \tag{B-1}$$

where $f(x)$ is the unknown attenuation function whose value is to be determined in the interval $0 \le s \le x$. The function $g(x)$ and $K(x, s)$ are known, and c is a constant. If $g(x)$ is bounded and continuous and if

$$\int_{0}^{x} |K(x, s)|\, ds \le L < \infty \tag{B-2}$$

then the solution will be unique and continuous (Wagner, 1953). This integral equation can be solved by a stepwise calculation that divides the interval x into subintervals of arbitrary width.

That is, consider the subdivision

$$f(x_n) = W(x_n) - (i/\lambda)^{\frac{1}{2}} \left\{ \int_{0}^{x_1} f(s)\, K(x_n, s)\, ds + \int_{x_1}^{x_2} f(s)\, K(x_n, s)\, ds + \cdots + \int_{x_{n-1}}^{x_n} f(s) K(x_n, s) ds \right\} \tag{B-3}$$

The unknown function, $f(s)$, is fitted with a polynomial of the form

$$f(s) = a_0 + a_1 s + a_2 s^2 \tag{B-4}$$

Increasing the degree of the polynomial to 3 or higher would result in even higher accuracy; however, the algebra becomes more complicated and sufficient accuracy can be obtained with the polynomial of degree 2. In some examples, the solution may become unstable for the higher degree polynomial and oscillate between the fitted points.

The solution of the integral equation requires special starting procedures. We suggest that the interpolating polynomial be of the form

$$f(s) = \alpha_0 + \alpha_1 s^{\frac{1}{2}} + \alpha_2 s + \alpha_3 s^{3/2} \quad , \quad 0 \le s \le x_3 \qquad \text{(B-5)}$$

and to use (B-4) for $x_2 \le s \le x_n$. The choice in (B-5) is a logical one if we assume the terrain is flat in the immediate vicinity of the transmitting antenna. If the terrain is flat the exact answer for the attenuation function is then in fact a half-order power series in the numerical distance. Requiring the polynomial in (B-5) to pass through the first four consecutive points yields

$$\alpha_0 = 1.0 \qquad \text{(B-6a)}$$

$$\alpha_1 = R_1 f(x_1) + R_2 f(x_2) + R_3 f(x_3) + R_4 \qquad \text{(B-6b)}$$

$$\alpha_2 = R_5 f(x_1) + R_6 f(x_2) + R_7 f(x_3) + R_8 \qquad \text{(B-6c)}$$

$$\alpha_3 = R_9 f(x_1) + R_{10} f(x_2) + R_{11} f(x_3) + R_{12} \qquad \text{(B-6d)}$$

The constants in (B-4) are found by requiring the polynomial to pass through the points x_{i-2}, x_{i-1} and x_i. It is a simple exercise to show that

$$a_0 = R_{13} f(x_i) + R_{14} f(x_{i-1}) + R_{15} f(x_{i-2}) \qquad \text{(B-7a)}$$

$$a_1 = R_{16} f(x_i) + R_{17} f(x_{i-1}) + R_{18} f(x_{i-2}) \qquad \text{(B-7b)}$$

$$a_2 = R_{19} f(x_i) + R_{20} f(x_{i-1}) + R_{21} f(x_{i-2}) \qquad \text{(B-7c)}$$

where the R's in (B-6) and (B-7) are defined as

$$D = (x_1 x_2 x_3)^{\frac{1}{2}} \left[x_1 (x_3^{\frac{1}{2}} - x_2^{\frac{1}{2}}) + x_2 (x_1^{\frac{1}{2}} - x_3^{\frac{1}{2}}) + x_3 (x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}}) \right] \tag{B-8a}$$

$$R_1 = x_2 x_3 (x_3^{\frac{1}{2}} - x_2^{\frac{1}{2}}) / D \tag{B-8b}$$

$$R_2 = x_1 x_3 (x_1^{\frac{1}{2}} - x_3^{\frac{1}{2}}) / D \tag{B-8c}$$

$$R_3 = x_1 x_2 (x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}}) / D \tag{B-8d}$$

$$R_4 = \left[x_1 (x_3^{3/2} - x_2^{3/2}) + x_2 (x_1^{3/2} - x_3^{3/2}) + x_3 (x_2^{3/2} - x_1^{3/2}) \right] / D \tag{B-8e}$$

$$R_5 = (x_2 x_3)^{\frac{1}{2}} (x_2 - x_3) / D \tag{B-8f}$$

$$R_6 = (x_1 x_3)^{\frac{1}{2}} (x_3 - x_1) / D \tag{B-8g}$$

$$R_7 = (x_1 x_2)^{\frac{1}{2}} (x_1 - x_2) / D \tag{B-8h}$$

$$R_8 = \left[x_1^{\frac{1}{2}} (x_2^{3/2} - x_3^{3/2}) + x_2^{\frac{1}{2}} (x_3^{3/2} - x_1^{3/2}) + x_3^{\frac{1}{2}} (x_1^{3/2} - x_2^{3/2}) \right] / D \tag{B-8i}$$

$$R_9 = (x_2 x_3)^{\frac{1}{2}} (x_3^{\frac{1}{2}} - x_2^{\frac{1}{2}}) / D \tag{B-8j}$$

$$R_{10} = (x_1 x_3)^{\frac{1}{2}} (x_1^{\frac{1}{2}} - x_3^{\frac{1}{2}}) / D \tag{B-8k}$$

$$R_{11} = (x_1 x_2)^{\frac{1}{2}} (x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}}) / D \tag{B-8ℓ}$$

$$R_{12} = \left[x_1^{\frac{1}{2}} (x_3 - x_2) + x_2^{\frac{1}{2}} (x_1 - x_3) + x_3^{\frac{1}{2}} (x_2 - x_1) \right] / D \tag{B-8m}$$

$$D_i = (x_{i-2} - x_{i-1}) \left[x_i^2 - x_i (x_{i-2} + x_{i-1}) + x_{i-2} x_{i-1} \right] \tag{B-8n}$$

$$R_{13} = x_{i-1} x_{i-2} (x_{i-2} - x_{i-1}) / D_i \tag{B-8o}$$

$$R_{14} = x_i x_{i-2} (x_i - x_{i-2}) / D_i \tag{B-8p}$$

$$R_{15} = x_i x_{i-1} (x_{i-1} - x_i) / D_i \tag{B-8q}$$

$$R_{16} = (x_{i-1}^2 - x_{i-2}^2) / D_i \tag{B-8r}$$

$$R_{17} = (x_{i-2}^2 - x_i^2) / D_i \tag{B-8s}$$

$$R_{18} = (x_i^2 - x_{i-1}^2) / D_i \tag{B-8t}$$

$$R_{19} = (x_{i-2} - x_{i-1}) / D_i \tag{B-8u}$$

$$R_{20} = (x_i - x_{i-2})/D_i \tag{B-8v}$$

$$R_{21} = (x_{i-1} - x_i)/D_i \tag{B-8w}$$

Using our polynomial interpolation formulas for $f(s)$ we find that the integrals in (B-3) all have the following generic form

$$p_\ell\,(x_i\,,\, x_j,\, x_k) = \int_{x_k}^{x_j} s^{\ell/2}\, K(x_i\,,\, s)\, ds \tag{B-9}$$

with

$$\begin{aligned} &0 \le k \le j-1 \\ &1 \le j \le i \\ &2 \le i \le n \\ &\ell = 0,\ 1,\ 2,\ 3,\ 4. \end{aligned} \tag{B-10}$$

These integrals are evaluated numerically using a five point Gaussian integration formula with special attention given to those integrals having singularities at either of the endpoints of integration.

Substituting (B-4) through (B-10) into (B-3) yields the following general expression for $f(x)$ at the i^{th} point

$$f(i)\left\{1 + (i/\lambda)^{\frac{1}{2}}\left[R_{13}\,(i)p_o(i,i,i-1) + R_{16}\,(i)p_2\,(i,i,i-1) + R_{19}\,(i)p_4\,(i,i,i-1)\right]\right\}$$

$$= W(i) - (i/\lambda)^{\frac{1}{2}}\left\{\sum_{j=1}^{3} p_0\,(i,j,j-1) + R_4 \sum_{j=1}^{3} p_1\,(i,j,j-1) + R_8 \sum_{j=1}^{3} p_2\,(i,j,j-1)\right.$$

$$+R_{12} \sum_{j=1}^{3} p_3\,(i,j,j-1) + f(1)\left[R_1 \sum_{j=1}^{3} p_1\,(i,j,j-1) + R_5 \sum_{j=1}^{3} p_2\,(i,j,j-1)\right.$$

$$+R_9 \sum_{j=1}^{3} p_3(i,j,j-1)\Big] + f(2)\Big[R_2 \sum_{j=1}^{3} p_1(i,j,j-1) + R_6 \sum_{j=1}^{3} p_2(i,j,j-1)$$

$$+R_{10} \sum_{j=1}^{3} p_3(i,j,j-1)\ R_{15}(4)p_0(i,4,3) + R_{18}(4)p_2(i,4,3) + R_{21}(4)p_4(i,4,3)\Big]$$

$$+ f(3)\Big[R_3 \sum_{j=1}^{3} p_1(i,j,j-1) + R_7 \sum_{j=1}^{3} p_2(i,j,j-1)\ + R_{11} \sum_{j=1}^{3} p_3(i,j,j-1)$$

$$+R_{14}(4)p_0(i,4,3) + R_{17}(4)p_2(i,4,3) + R_{20}(4)p_4(i,4,3) + R_{15}(5)p_0(i,5,4)$$

$$+R_{18}(5)p_2(i,5,4) + R_{21}(5)p_4(i,5,4)\Big] + \sum_{m=4}^{i-2} f(m)\Big[R_{13}(m)p_0(i,m,m-1)$$

$$+R_{14}(m+1)p_0(i,m+1,m) + R_{15}(m+2)p_0(i,m+2,m+1)$$

$$+R_{16}(m)p_2(i,m,m-1) + R_{17}(m+1)p_2(i,m+1,m) + R_{18}(m+2)p_2(i,m+2,m+1)$$

$$+R_{19}(m)p_4(i,m,m-1)+R_{20}(m+1)p_4(i,m+1,m)+R_{21}(m+2)p_4(i,m+2,m+1)\Big]$$

$$+f(i-1)\Big[R_{13}(i-1)p_0(i,i-1\ i-2)+R_{14}(i)p_0(i,i,i-1)+R_{16}(i-1)p_2(i,i-1,i-2)$$

$$+R_{17}(i)p_2(i,i,i-1)+R_{19}(i-1)p_4(i,i-1,i-2)+R_{20}(i)p_4(i,i,i-1)\Big]\Big\} \qquad \text{(B-11)}$$

Reference

(B-1) Wagner, Carl (1953), "On the numerical solution of Volterra integral equations," J. Math. and Phys. 32, pp. 289-401.

APPENDIX C: The Computer Program and Flow Chart

Program Wagner implements the procedure given in Appendix B for solving the integral equation derived in Appendix A. Flexibility is obtained by using appropriate versions of three subroutines:

(1) TERRANE, which returns the height, slope, and ground constants (σ, ϵ_r) as a function of distance, x. By writing appropriate statements in this subroutine the user can define any propagation path he needs. The general form of the subroutine TERRANE is shown on page 49, and two particular implementations used for examples in this report are listed on pages 50 and 51 .

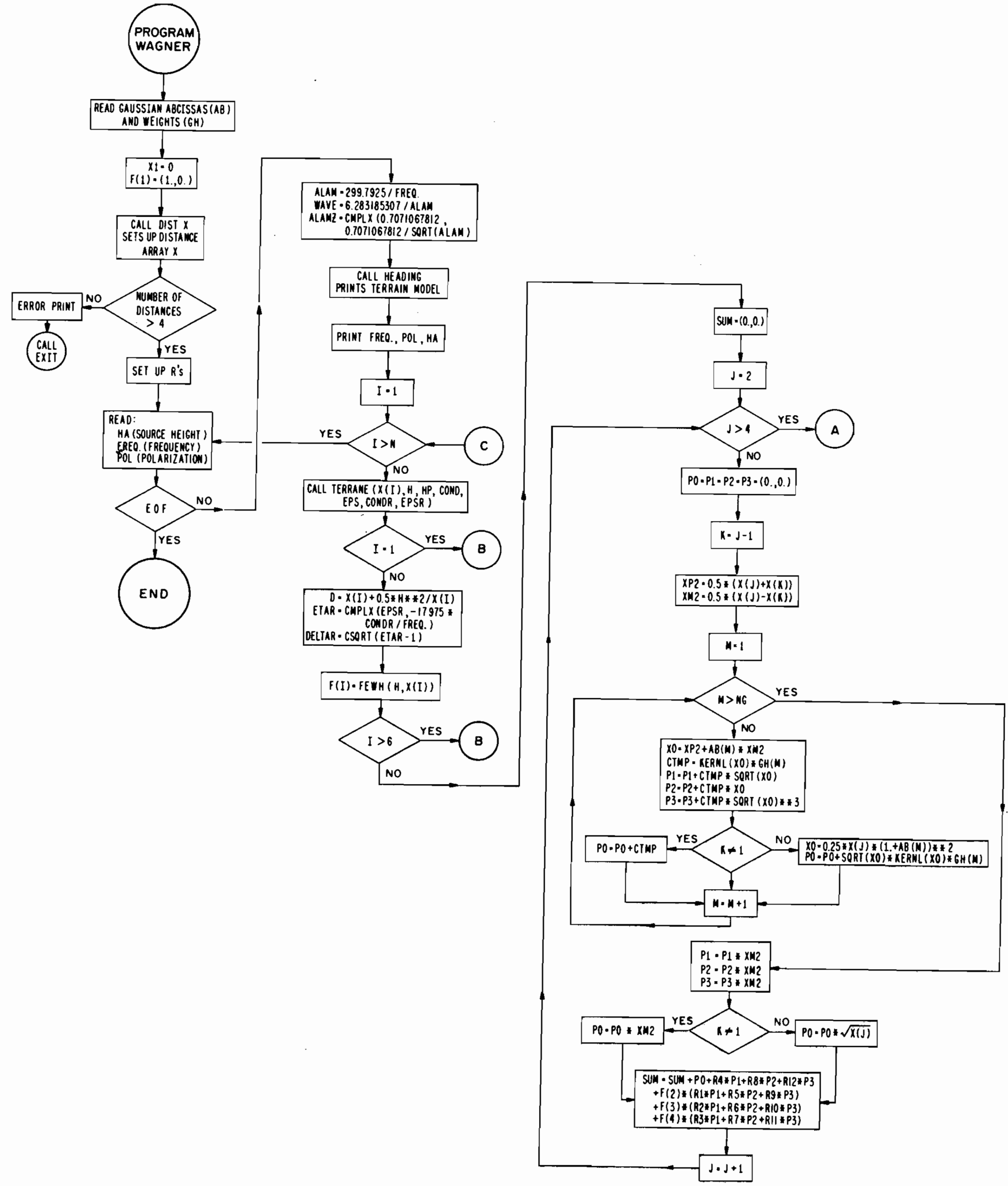
(2) DISTX, which returns the set of distances $x(I)$ at which the function $F(x)$ will be calculated. The general form of DISTX is shown on page 45, and two particular implementations are shown on pages 46 and 47 .

(3) KERNL, which computes the kernel of the integral equation. Program Wagner can be used to solve other integral equations of the form (B-1) if the kernel includes the factor $[s(x-s)]^{-\frac{1}{2}}$ by modifying subroutine KERNL. For example, WAGNER can solve Hufford's integral equation.

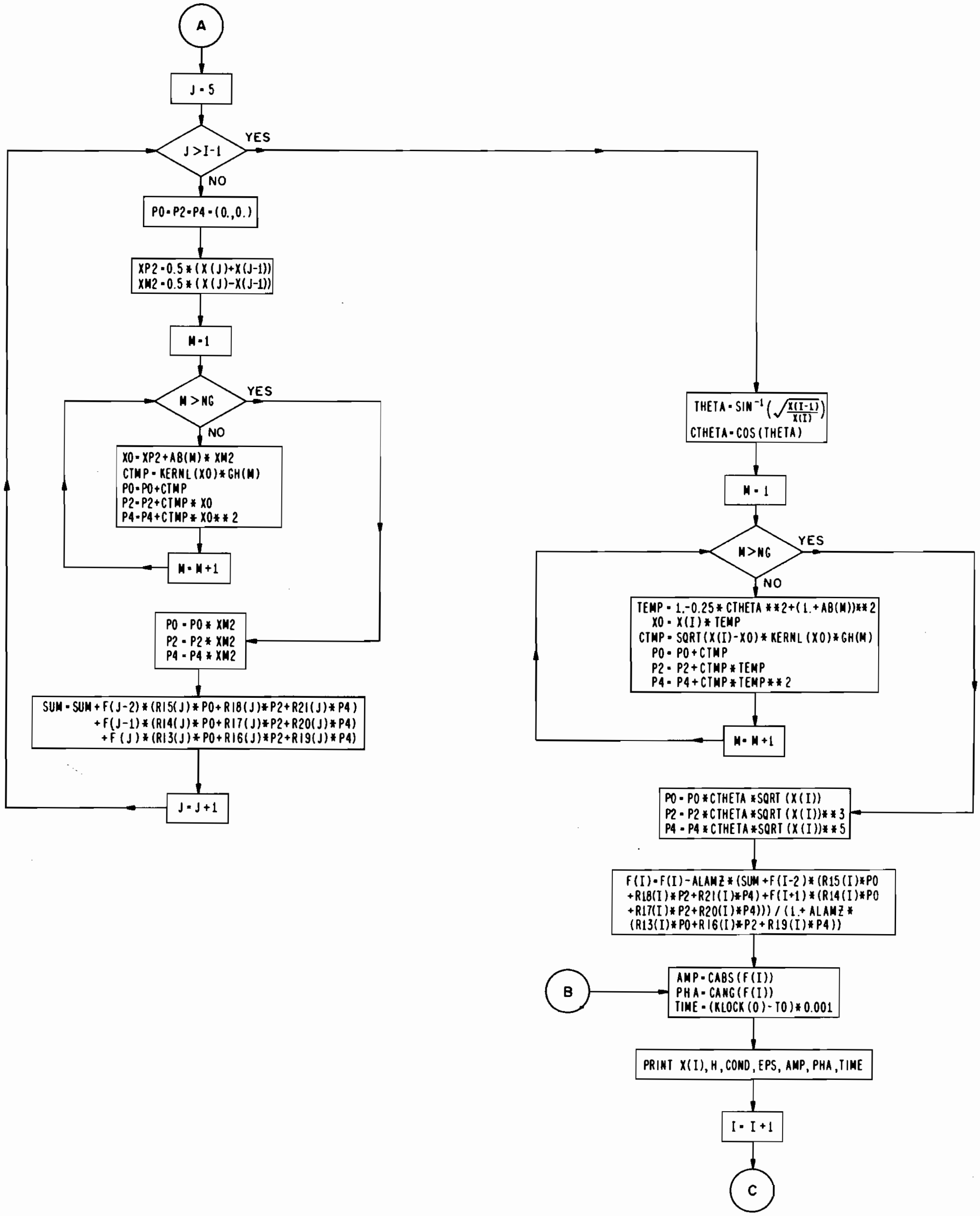
Comment cards in the listings that follow explain the program's operation. The input card sequence for Program WAGNER is

Card	Cols.	Description
1	1-10	The number of Gaussian quadrature abcissas and weights (5 recommended)
2 through 4	3-33 & 36-66	Values for the Gaussian weights and abcissas.
5 through N+4	1-10	The N points at which the attenuation function is to be calculated. These distances are read in kilometers, by DISTX.
N + 5		A blank card which signals the end of the distance deck when the form of DISTX is that given on page 47. When DISTX takes the form given on page 46, no blank card is required.
N + 6	1-10	Source height in kilometers.
	11-20	Frequency in Megahertz.
	21-30	Polarization, 1. = vertical, 2. = horizontal.

Following is a flow chart together with a statement listing (Fortran 3800) of the computer program, and a sample output.



Flow chart for computer program



Flow chart for computer program

```
      PROGRAM WAGNER
C
C        A PROGRAM TO COMPUTE HF GROUND WAVE ATTENUATION
C        IRREGULAR, INHOMOGENEOUS TERRAIN.  REFERENCE:
C        TELECOMMUNICATIONS RESEARCH REPORT, No. 7 , 1970.
C
      DIMENSION IPOL(2)
      COMMON /0/ F(2000),R13(2000),R14(2000),R15(2000),R16(2000),
     1 R17(2000),R18(2000),R19(2000),R20(2000),R21(2000)
      COMMON /1/ HA
      COMMON /2/ D,H,HP
      COMMON /3/ DELTAR,WAVE
      COMMON /4/ FREQ,POL
      COMMON /5/ NG,AB(48),GH(48)
      COMMON /6/ N,X(2001),I
      TYPE DOUBLE DAB,DGH
      COMPLEX FEWH,F,ALAMZ,SUM,DELTAR,ETAR
      COMPLEX KERNL,P0,P1,P2,P3,P4,CTMP
      IPOL(1)=8H  VERTIC  $  IPOL(2)=8HHORIZONT
C
C           READ GAUSSIAN QUADRATURE ABCISSAS AND WEIGHTS
C
      READ 1000, NG
 1000 FORMAT(I10)
      NR=(NG+1)/2
      DO 1 L=1,NR
      READ 1010, DAB,DGH
 1010 FORMAT(2D33.25)
      J=NG-L+1
      AB(L)=DAB
      AB(J)=-AB(L)
      GH(L)=DGH
    1 GH(J)=GH(L)
C
C           CALL SUBROUTINE TO SET UP DISTANCE ARRAY X IN METERS
C           START WITH X(2).  X(1)=0. HAS ALREADY BEEN SET.
C           THE DISTANCES DO NOT HAVE TO BE EQUALLY SPACED.
C           SUBROUTINE DISTX SHOULD MAKE SURE N < 2000
C
      X(1)=0.
      F(1)=(1.,0.)
      CALL DISTX
C
C           MAKE SURE THERE ARE AT LEAST 4 DISTANCES
      IF (N.GE.4) GO TO 4
      PRINT 1040
 1040 FORMAT (*0NUMBER OF DISTANCES 0 4*)
      CALL EXIT
C
    4 SQRTX2=SQRT(X(2))
      SQRTX3=SQRT(X(3))
      SQRTX4=SQRT(X(4))
      D1=SQRT(X(2)*X(3)*X(4))*(X(2)*(SQRTX4-SQRTX3)+X(3)*(SQRTX2-SQRTX4)
     1 +X(4)*(SQRTX3-SQRTX2))
      R1=X(3)*X(4)*(SQRTX4-SQRTX3)/D1
      R2=X(2)*X(4)*(SQRTX2-SQRTX4)/D1
      R3=X(2)*X(3)*(SQRTX3-SQRTX2)/D1
      R4=(X(2)*(SQRTX4**3-SQRTX3**3)+X(3)*(SQRTX2**3-SQRTX4**3)
     1 +X(4)*(SQRTX3**3-SQRTX2**3))/D1
```

```
      R5=SQRT(X(3)*X(4))*(X(3)-X(4))/D1
      R6=SQRT(X(2)*X(4))*(X(4)-X(2))/D1
      R7=SQRT(X(2)*X(3))*(X(2)-X(3))/D1
      R8=(SQRTX2*(SQRTX3**3-SQRTX4**3)+SQRTX3*(SQRTX4**3-SQRTX2**3)
     1 +SQRTX4*(SQRTX2**3-SQRTX3**3))/D1
      R9=SQRT(X(3)*X(4))*(SQRTX4-SQRTX3)/D1
      R10=SQRT(X(2)*X(4))*(SQRTX2-SQRTX4)/D1
      R11=SQRT(X(2)*X(3))*(SQRTX3-SQRTX2)/D1
      R12=(SQRTX2*(X(4)-X(3))+SQRTX3*(X(2)-X(4))+SQRTX4*(X(3)-X(2)))/D1
      DO 10 M=5,N
      M1=M-1
      M2=M-2
      D2=(X(M2)-X(M1))*(X(M)**2-X(M)*(X(M1)+X(M2))+X(M1)*X(M2))
      R13(M)=X(M1)*X(M2)*(X(M2)-X(M1))/D2
      R14(M)=X(M)*X(M2)*(X(M)-X(M2))/D2
      R15(M)=X(M)*X(M1)*(X(M1)-X(M))/D2
      R16(M)=(X(M1)**2-X(M2)**2)/D2
      R17(M)=(X(M2)**2-X(M)**2)/D2
      R18(M)=(X(M)**2-X(M1)**2)/D2
      R19(M)=(X(M2)-X(M1))/D2
      R20(M)=(X(M)-X(M2))/D2
   10 R21(M)=(X(M1)-X(M))/D2
C
C            READ SOURCE HEIGHT, FREQUENCY, AND POLARIZATION
C     COL        DESCRIPTION
C     1-10     SOURCE HEIGHT, KM
C    11-20     FREQUENCY, MHZ
C    21-30     POLARIZATION, 1. = VERTICAL, 2. = HORIZONTAL
C
   20 READ 2000, HA,FREQ,POL
 2000 FORMAT (3F10.4)
      IF (EOF,60) 999,22
   22 HA=HA*1.E3
      KPOL=POL
      ALAM=2.997925E2/FREQ
      WAVE=6.283185307/ALAM
      ALAMZ = ((0.7071067812,0.7071067812)/SQRTF(ALAM))
      CALL HEADING
      PRINT 2500, FREQ,IPOL(KPOL),HA
 2500 FORMAT (*0FREQUENCY =*,F10.2,10X,A8,*AL POLARIZATION*,10X,*ANTENNA
     1 HEIGHT =*,F6.2,* METERS*//
     2 9X,*X*,14X,*Z*,10X,*CONDUCTIVITY*,3X,*DIELECTRIC*,15X,*F(X)*,22X,
     3 *TIMING*/8X,*(M)*,12X,*(M)*,12X,*(MHO/M)*,6X,*CONSTANT*,8X,*MAG*,
     4 13X,*ARG*,16X,*(SEC)*)
      T0=KLOCK(0)
C
C            LOOP ON DISTANCE
C
      DO 100 I=1,N
      CALL TERRANE (X(I),H,HP,COND,EPS,CONDR,EPSR)
      IF (I.EQ.1) GO TO 75
      D=X(I)+(H**2)/(2.*X(I))
      ETAR = CMPLX(EPSR,-17975.*CONDR/FREQ)
      DELTAR = CSQRT(ETAR - 1.)
```

```
      IF(KPOL.EQ.1) DELTAR = DELTAR/ETAR
      F(I)=FEWH(H,X(I))
      IF (I.LE.6) GO TO 75
C
C                J = 2 THROUGH 4
C
      SUM=(0.,0.)
      DO 40 J=2,4
      P0=P1=P2=P3=(0.,0.)
      K=J-1
      XP2=0.5*(X(J)+X(K))
      XM2=0.5*(X(J)-X(K))
      DO 35 M=1,NG
      X0=XP2+AB(M)*XM2
      CTMP=KERNL(X0)*GH(M)
      P1=P1+CTMP*SQRT(X0)
      P2=P2+CTMP*X0
      P3=P3+CTMP*SQRT(X0)**3
      IF (K.NE.1) GO TO 33
      X0=0.25*X(J)*(1.+AB(M))**2
      P0=P0+SQRT(X0)*KERNL(X0)*GH(M)
      GO TO 35
   33 P0=P0+CTMP
   35 CONTINUE
      P1=P1*XM2
      P2=P2*XM2
      P3=P3*XM2
      IF (K.NE.1) GO TO 38
      P0=P0*SQRT(X(J))
      GO TO 40
   38 P0=P0*XM2
   40 SUM=SUM+P0+R4*P1+R8*P2+R12*P3 +F(2)*(R1*P1+R5*P2+R9*P3)
     1     +F(3)*(R2*P1+R6*P2+R10*P3)+F(4)*(R3*P1+R7*P2+R11*P3)
C
C                J = 5 THROUGH I-1
C
      I1=I-1
      DO 50 J=5,I1
      P0=P2=P4=(0.,0.)
      XP2=0.5*(X(J)+X(J-1))
      XM2=0.5*(X(J)-X(J-1))
      DO 45 M=1,NG
      X0=XP2+AB(M)*XM2
      CTMP=KERNL(X0)*GH(M)
      P0=P0+CTMP
      P2=P2+CTMP*X0
   45 P4=P4+CTMP*X0**2
      P0=P0*XM2
      P2=P2*XM2
      P4=P4*XM2
   50 SUM=SUM+F(J-2)*(R15(J)*P0+R18(J)*P2+R21(J)*P4)
     1         +F(J-1)*(R14(J)*P0+R17(J)*P2+R20(J)*P4)
     2         +F(J)   *(R13(J)*P0+R16(J)*P2+R19(J)*P4)
C
```

```
C                    J=I
C
      THETA=ASINF(SQRT(X(I1)/X(I)))
      CTHETA=COSF(THETA)
      P0=P2=P4=(0.,0.)
      DO 55 M=1,NG
      TEMP=1.-0.25*CTHETA**2*(1.+AB(M))**2
      X0=X(I)*TEMP
      CTMP=SQRT(X(I)-X0)*KERNL(X0)*GH(M)
      P0=P0+CTMP
      P2=P2+CTMP*TEMP
   55 P4=P4+CTMP*TEMP**2
      P0=P0*CTHETA*SQRT(X(I))
      P2=P2*CTHETA*SQRT(X(I))**3
      P4=P4*CTHETA*SQRT(X(I))**5
C
C     EQUATION (B-11)
C
      F(I)=(F(I)-ALAMZ*(SUM+F(I-2)*(R15(I)*P0+R18(I)*P2+R21(I)*P4)
     1 +F(I1)*(R14(I)*P0+R17(I)*P2+R20(I)*P4)))/(1.+ALAMZ*(R13(I)*P0
     2 +R16(I)*P2+R19(I)*P4))
   75 AMP  =  CABS(F(I))
      PHA  =  CANG(F(I))
      TIME=(KLOCK(0)-T0)*0.001
      PRINT 8000, X(I),H,COND,EPS,AMP,PHA,TIME
 8000 FORMAT (*0*,F12.2,F18.9,F14.6,F13.4,E18.8,E16.8,F15.3)
  100 CONTINUE
C
      GO TO 20
  999 CALL EXIT
         END
```

```
      SUBROUTINE DISTX
C          READ DISTANCES IN KM AND CONVERTS THEM TO METERS
C          (A DISTANCE OF ZERO SIGNALS END OF DISTANCE DECK)
      COMMON /6/ N,X(2001),I
C
C     IN THIS SUBROUTINE THE USER MUST FILL
C     THE X(I) ARRAY WITH N VARIABLES.
C
      RETURN
      END
```

```
      SUBROUTINE DISTX
C           COMPUTES EQUALLY SPACED DISTANCES
      COMMON /6/ N,X(2001),I
C                INPUT
C           DMIN -- FIRST DISTANCE IN KM
C           DMAX -- MAXIMUM DISTANCE IN KM
C           DINC -- INCREMENT ON DISTANCE IN KM
C
      READ 1000, DMIN,DMAX,DINC
 1000 FORMAT (3F10.2)
      IF (DMIN.EQ.0.) DMIN=DMIN+DINC
      N=(DMAX-DMIN)/DINC+2
      DO 10 I=2,N
      X(I)=(DMIN+(I-2)*DINC)*1.E3
   10 CONTINUE
      RETURN
         END
```

Note, this is an example of subroutine DISTX.

```
      SUBROUTINE DISTX
C           READ DISTANCES IN KM AND CONVERTS THEM TO METERS
C           (A DISTANCE OF ZERO SIGNALS END OF DISTANCE DECK)
      COMMON /6/ N,X(2001),I
      DO 2 I=2,2001
      READ 1020, X(I)
 1020 FORMAT (F10.5)
      IF (X(I).EQ.0.) GO TO 3
      X(I)=X(I)*1.E+3
    2 CONTINUE
      PRINT 1030
 1030 FORMAT (*0NUMBER OF DISTANCES EXCEEDS DIMENSION*)
      CALL EXIT
    3 N=I-1
      END
```

Note, this is an example of subroutine DISTX.

```
      FUNCTION KERNL(X0)
C
C      SUBROUTINE OF WAGNER.  COMPUTES
C      KERNEL OF INTEGRAL EQUATION.  SEE
C      EQ. (A-13).
C
      COMMON /1/ HA
      COMMON /2/ D,H,HP
      COMMON /3/ DELTAR,WAVE
      COMMON /4/ FREQ,POL
      COMMON /5/ NG,AB(48),GH(48)
      COMMON /6/ NX,X(2001),I
      COMPLEX KERNL,FEWH,DELTA,DELTAR,ETA
      CALL TERRANE(X0,H0,HP0,COND,EPS,CONDR,EPSR)
      ETA=CMPLX(EPS,-17975.*COND/FREQ)
      DELTA=CSQRT(ETA-1.)
      IF(POL.EQ.1.) DELTA=DELTA/ETA
      XMS=X(I)-X0
      HD=H-H0
      R1=SQRT(X0**2+HA**2)
      RW = WAVE*(X0 + ((H0**2)/(2.*X0)) + XMS + ((HD**2)/(2.*XMS)) - D)
      KERNL=CMPLX(COSF(RW),-SINF(RW))*SQRT(X(I)/(R1*XMS))*((HP0+DELTA
     1 -DELTAR)*FEWH(HD,XMS) - (HD/XMS))
C
C      THE FACTOR (DELTA-DELTAR) ARISES IN
C      MIXED-PATH PROBLEMS.
C
      RETURN
      END
```

```
      SUBROUTINE TERRANE (X,H,HP,COND,EPS,CONDR,EPSR)
C             SUBROUTINE FOR WAGNER. DEFINES TERRAIN, PROFILE AND
                                     GROUND CONSTANTS.
C
C     INPUT IS DISTANCE X IN METERS.
C     OUTPUT IS TERRAIN HEIGHT, H, SLOPE, HP,
C     GROUND CONSTANTS, CONDR, EPSR, COND, EPS.
C     IN MIXED PATH CALCULATIONS, CONOR AND EPSR
C     ARE RELATIVE VALUES FOR σ AND ε_r .
C     THEY ARE USED TO COMPUTE
C     DELTAR IN PROGRAM WAGNER.
C     IN FUNCTION KERNL THE DIFFERENCE
C     (DELTA-DELTAR) IS
C     COMPUTED. THIS DIFFERENCE TAKES INTO
C     ACCOUNT CHANGES
C     IN σ AND ε_r WITH DISTANCE.
C     CONDR AND EPSR ARE USUALLY
C     TAKEN TO BE THE VALUES OF
C     σ AND ε_r FOR THE FIRST
C     SECTION OF PATH.
C

C
C     IN THIS SUBROUTINE THE USER MUST DEFINE THE FOLLOWING
C     VARIABLES
C     H =
C     HP =
C     CONDR =
C     EPSR =
C     COND =
C     EPS =
C
C           PRINT HEADING
      ENTRY HEADING
      PRINT 50,A
   50 FORMAT (*A SMOOTH SPHERE WITH RADIUS*,E12.3)
      RETURN
      END
```

```
      SUBROUTINE TERRANE (X,H,HP,COND,EPS,CONDR,EPSR)
C             SUBROUTINE FOR WAGNER.   DEFINES TERRAIN.
C      SMOOTH SPHERE
      COMMON /1/ HA
      DATA (A=8.5E6)
C
C        COMPUTE HEIGHT,SLOPE,CONDUCTIVITY AND DIELECTRIC CONSTANT AT X
      HP=-X/A
      H=.5*X*HP- HA
      CONDR = .01
      EPSR = 10.
      COND = .01
      EPS = 10.
      RETURN
C
C          PRINT HEADING
      ENTRY HEADING
      PRINT 50,A
   50 FORMAT (*A SMOOTH SPHERE WITH RADIUS*,E12.3)
      RETURN
         END
```

Note, this is an example of subroutine TERRANE.

```
      SUBROUTINE TERRANE (X,H,HP,COND,EPS,CONDR,EPSR)
C               SUBROUTINE FOR WAGNER.   DEFINES TERRAIN.
C TABLE MOUNTAIN PATH WITH KBOL AS TRANSMITTER
      COMMON /1/ HA
C
C        COMPUTE HEIGHT,SLOPE,CONDUCTIVITY AND DIELECTRIC CONSTANT AT X
      H = 50.*TANHF((X-5000.)/100.)+50.-HA
      HP=0.5*(1.-(TANHF((X-5000.)/100.))**2)
      CONDR = .01
      EPSR = 10.
C
C        A FOUR SECTION PATH
C
      X1 = 28574.0
      X2 = 35000.
      X3 = 45000.0
      IF(X.GT.X1.AND.X.LE.X2) GO TO 20
      IF(X.GT.X2.AND.X.LE.X3) GO TO 30
      IF(X.GT.X3) GO TO 40
      COND = .01
      EPS = 10.
      GO TO 10
   20 COND =  2.0
      EPS =  .81
      GO TO 10
   30 COND = .01
      EPS = 10.
      GO TO 10
   40 COND = 2.0
      EPS =  81.0
   10 CONTINUE
      RETURN
C
C           PRINT HEADING
      ENTRY HEADING
      PRINT 50
   50 FORMAT(*TABLE MOUTAIN PATH WITH KBOL AS TRANSMITTER*)
      RETURN
         END
```

Note, this is an example of subroutine TERRANE.

```
      COMPLEX FUNCTION FEWH(HD,XD)
C
C       THE ATTENUATION FUNCTION,
C       EQ (A-13), OF TELECOMMUNICATIONS RESEARCH
C       REPORT No. 7 , 1970. INPUT IS THE
C       HEIGHT HD AND THE DISTANCE
C       XD.
C
      COMMON /3/ DELTAR,WAVE
      COMPLEX FEWH,TEMP,Q,Z,Z2,ZZ,HWERF,WERFZ,WERF,ZWERF,DELTAR
      TEMP=(0.7071067812,-0.7071067812)*SQRT(.5*WAVE)
      XD2=SQRT(XD)
      Q=-TEMP*HD/XD2
      Z=TEMP*DELTAR*XD2 + Q
      ZZ=-Z
      ZI=AIMAG(ZZ)
      IF (ZI.LT.0..OR.(ABS(REAL(ZZ)).LT.6..AND.ZI.LT.6.)) GO TO 10
      Z2=ZZ**2
      HWERF=(Z2-2.)/(ZZ*(Z2-3.5))
      GO TO 12
   10 WERFZ=WERF(ZZ)
      HWERF=ZZ-0.5*WERFZ/(ZZ*WERFZ+(0.,-0.56418958))
   12 ZWERF=Z+HWERF
      FEWH=(Q*ZWERF-0.5)/(Z*ZWERF-0.5)
      RETURN
         END
```

```
      COMPLEX FUNCTION WERF(ZZZ)
C
C       THE FUNCTION w(z),
C       ABRAMOWITZ AND STEGUN, 1964)
C       WRITTEN BY DR. GEORGE HUFFORD, AND MODIFIED BY
C       DR. R. H. OTT
C
      COMPLEX Z,ZZZ,ZV,V,Z2,C,W,S
      DIMENSION C(12),W(5,4)
      EQUIVALENCE (S,C(12))
      DATA (C(1) = (.0,-.5641895835))
      DATA (((W(I,J),I=1,5),J=1,4)=(1.,.0),
     X (3.678794411714423E-01,6.071577058413937E-01),
     X (1.831563888873418E-02,3.400262170660662E-01),
     X (1.234098040866788E-04,2.011573170376004E-01),
     X (1.125351747192646E-07,1.459535899001528E-01),
     X (4.275835761558070E-01,0.000000000000000E+00),
     X (3.047442052569126E-01,2.082189382028316E-01),
     X (1.402395813662779E-01,2.222134401798991E-01),
     X (6.531777728904697E-02,1.739183154163490E-01),
     X (3.628145648998864E-02,1.358389510006551E-01),
     X (2.553956763105058E-01,0.000000000000000E+00),
     X (2.184926152748907E-01,9.299780939260186E-02),
     X (1.479527595120158E-01,1.311797170842178E-01),
     X (9.271076642644332E-02,1.283169622282615E-01),
     X (5.968692961044590E-02,1.132100561244882E-01),
     X (1.790011511813930E-01,0.000000000000000E+00),
     X (1.642611363929861E-01,5.019713513524966E-02),
     X (1.307574696698522E-01,8.111265047745472E-02),
     X (9.640250558304439E-02,9.123632600421258E-02),
     X (6.979096164964750E-02,8.934000024036461E-02))
      XX=REAL(ZZZ)
      YY=AIMAG(ZZZ)
      X=ABSF(XX)
      Y=ABSF(YY)
      Z=CMPLX(X,Y)
      LZ2=0
      IF(X.GE.4.5.OR.Y.GE.3.5) GO TO 100


C
C       CONVERGING SERIES
C
```

```
      I=X+.5
      J=Y+.5
      V=CMPLX(FLOATF(I),FLOAT(J))
      ZV=Z-V
      C(2)=W(I+1,J+1)
      AI=0.
      DO 10 I=3,12
      AI=AI-.5
      C(I)=(V*C(I-1)+C(I-2))/AI
   10 CONTINUE
      J=12
      DO 11 I=2,11
      J=J-1
   11 S=S*ZV+C(J)
   20 IF(YY.GE.0.) GO TO 30
      IF(.NOT.LZ2) Z2=Z*Z
      S=2.*CEXP(-Z2)-S
      IF(XX.GT.0.) S=CONJG(S)
      GO TO 200
   30 IF(XX.LT.0.) S=CONJG(S)
  200 WERF=S
      RETURN
  100 LZ2=1
      Z2=Z*Z

      C
      C       ASYMPTOTIC SERIES
      C


      S = Z*((0.,0.4613135279)/(Z2 - 0.1901635092) + (0.,0.09999216168)/
     X(Z2 - 1.7844927485) + (0.,0.0028838938748)/(Z2 - 5.5253437437 9))
      GO TO 20
      END
```

Input data for the case of a smooth cylindrical earth:

Card #1:	5 (Column 5)		
Card #2:	0.9061798459	0.2369268851	
Card #3:	0.5384693101	0.4786286704	
Card #4:	0.0000000000	0.56888888888	
Card #5 through 62:	1.0, 2.0, ..., 58.		
Card #63:	0.0 (column 8)	1.0 (column 18)	1.0 (column 28)

Following is the output from this example.

A SMOOTH SPHERE WITH RADIUS 8.500+006

FREQUENCY = 1.00 VERTICAL POLARIZATION ANTENNA HEIGHT = 0.00 METERS

X (M)	Z (M)	CONDUCTIVITY (MHO/M)	DIELECTRIC CONSTANT	F(X) MAG	F(X) ARG	TIMING (SEC)
0.00	0.000000000			1.00000000+000	0.00000000+000	0.000
1000.00	-0.058823529	0.010000	10.0000	9.62785706-001	-4.24608707-001	0.007
2000.00	-0.235294118	0.010000	10.0000	9.34092461-001	-5.97768976-001	0.033
3000.00	-0.529411765	0.010000	10.0000	9.07383363-001	-7.28975441-001	0.045
4000.00	-0.941176471	0.010000	10.0000	8.81991252-001	-8.38214925-001	0.056
5000.00	-1.470588235	0.010000	10.0000	8.57643738-001	-9.33250508-001	0.069
6000.00	-2.117647059	0.010000	10.0000	8.34825345-001	-1.02254934+000	0.348
7000.00	-2.882352941	0.010000	10.0000	8.12513417-001	-1.10098530+000	0.665
8000.00	-3.764705882	0.010000	10.0000	7.90993050-001	-1.17332944+000	1.022
9000.00	-4.764705882	0.010000	10.0000	7.70208750-001	-1.24066203+000	1.413
10000.00	-5.882352941	0.010000	10.0000	7.50115518-001	-1.30377135+000	1.838
11000.00	-7.117647059	0.010000	10.0000	7.30675486-001	-1.36325394+000	2.310
12000.00	-8.470588235	0.010000	10.0000	7.11855902-001	-1.41957439+000	2.819
13000.00	-9.941176471	0.010000	10.0000	6.93627838-001	-1.47310306+000	3.358
14000.00	-11.529411765	0.010000	10.0000	6.75965320-001	-1.52414101+000	3.942
15000.00	-13.235294118	0.010000	10.0000	6.58844727-001	-1.57293697+000	4.558
16000.00	-15.058823529	0.010000	10.0000	6.42244356-001	-1.61969937+000	5.216
17000.00	-17.000000000	0.010000	10.0000	6.26144100-001	-1.66460498+000	5.913
18000.00	-19.058823530	0.010000	10.0000	6.10525209-001	-1.70780530+000	6.649
19000.00	-21.235294118	0.010000	10.0000	5.95370101-001	-1.74943145+000	7.423
20000.00	-23.529411765	0.010000	10.0000	5.80662214-001	-1.78959780+000	8.231
21000.00	-25.941176470	0.010000	10.0000	5.66385892-001	-1.82840489+000	9.094
22000.00	-28.470588235	0.010000	10.0000	5.52526281-001	-1.86594171+000	9.982
23000.00	-31.117647058	0.010000	10.0000	5.39069260-001	-1.90228748+000	10.907
24000.00	-33.882352941	0.010000	10.0000	5.26001367-001	-1.93751312+000	11.869
25000.00	-36.764705882	0.010000	10.0000	5.13309747-001	-1.97168246+000	12.874

26000.00	-39.764705882	0.010000	10.0000	5.00982101-001	-2.00485323+000	13.926
27000.00	-42.882352942	0.010000	10.0000	4.89006649-001	-2.03707782+000	15.002
28000.00	-46.117647058	0.010000	10.0000	4.77372092-001	-2.06840402+000	16.113
29000.00	-49.470588235	0.010000	10.0000	4.66067577-001	-2.09887556+000	17.270
30000.00	-52.941176470	0.010000	10.0000	4.55082679-001	-2.12853262+000	18.462
31000.00	-56.529411764	0.010000	10.0000	4.44407365-001	-2.15741220+000	19.696
32000.00	-60.235294118	0.010000	10.0000	4.34031984-001	-2.18554855+000	20.971
33000.00	-64.058823530	0.010000	10.0000	4.23947239-001	-2.21297341+000	22.274
34000.00	-68.000000000	0.010000	10.0000	4.14144178-001	-2.23971633+000	23.624
35000.00	-72.058823530	0.010000	10.0000	4.04614181-001	-2.26580489+000	25.009
36000.00	-76.235294119	0.010000	10.0000	3.95348947-001	-2.29126493+000	26.429
37000.00	-80.529411763	0.010000	10.0000	3.86340488-001	-2.31612073+000	27.891
38000.00	-84.941176470	0.010000	10.0000	3.77579473-001	-2.34038399+000	29.399
39000.00	-89.470588237	0.010000	10.0000	3.69060501-001	-2.36409375+000	30.941
40000.00	-94.117647059	0.010000	10.0000	3.60775578-001	-2.38726194+000	32.518
41000.00	-98.882352941	0.010000	10.0000	3.52717656-001	-2.40990741+000	34.150
42000.00	-103.764705881	0.010000	10.0000	3.44879922-001	-2.43204799+000	35.803
43000.00	-108.764705881	0.010000	10.0000	3.37255794-001	-2.45370059+000	37.498
44000.00	-113.882352941	0.010000	10.0000	3.29838903-001	-2.47488123+000	39.216
45000.00	-119.117647059	0.010000	10.0000	3.22623091-001	-2.49560513+000	40.990
46000.00	-124.470588233	0.010000	10.0000	3.15602395-001	-2.51588681+000	42.806
47000.00	-129.941176470	0.010000	10.0000	3.08771045-001	-2.53574010+000	44.655
48000.00	-135.529411763	0.010000	10.0000	3.02123453-001	-2.55517817+000	46.549
49000.00	-141.235294119	0.010000	10.0000	2.95654207-001	-2.57421367+000	48.476
50000.00	-147.058823530	0.010000	10.0000	2.89358067-001	-2.59285867+000	50.431
51000.00	-153.000000000	0.010000	10.0000	2.83229953-001	-2.61112474+000	52.444
52000.00	-159.058823530	0.010000	10.0000	2.77264947-001	-2.62902301+000	54.481
53000.00	-165.235294115	0.010000	10.0000	2.71458281-001	-2.64656414+000	56.555
54000.00	-171.529411767	0.010000	10.0000	2.65805333-001	-2.66375842+000	58.664
55000.00	-177.941176470	0.010000	10.0000	2.60301624-001	-2.68061575+000	60.818
56000.00	-184.470588233	0.010000	10.0000	2.54942815-001	-2.69714566+000	63.016
57000.00	-191.117647056	0.010000	10.0000	2.49724697-001	-2.71335735+000	65.245
58000.00	-197.882352941	0.010000	10.0000	2.44643190-001	-2.72925972+000	67.516

GPO 832 – 966